The Standard Model of particle physics and beyond.

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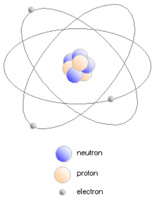
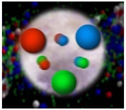
Outline.

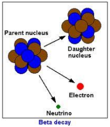
Building blocks describing matter.



- **Atom compositness.**
  * Neutrons.
  * Protons.
  * **Electrons**.



- **Proton and neutron compositness**.
  * Naively: **up and down quarks**.
  * In reality: dynamical objects made of
    - Valence and sea quarks.
    - Gluons [see below...].



- **Beta decays**.
  * $n \to p + e^- + \bar{\nu}_e$.
  * Needs for a **neutrino**.

Three families of fermionic particles [Why three?].

- **Quarks:**

Family	Up-type quark	Down-type quark
1st generation	up quark **u**	down quark **d**
2nd generation	charm quark **c**	strange quark **s**
3rd generation	top quark **t**	bottom quark **b**

- **Leptons:**

Family	Charged lepton	Neutrino
1st generation	electron $\mathbf{e}^-$	electron neutrino $\nu_{\mathbf{e}}$
2nd generation	muon μ^-	muon neutrino ν_μ
3rd generation	tau τ^-	tau neutrino ν_τ

- In addition, the associated **antiparticles.**
- The only difference between generations lies in the **(increasing) mass.**
- **Experimental status** [Particle Data Group Review].
  * **All** these particules have been observed.
  * Last ones: top quark (1995) and tau neutrino (2001).

Fundamental interactions and gauge bosons.

- **Electromagnetism**.
  - Interactions between charged particles (quarks and charged leptons).
  - Mediated by **massless photons** γ (spin one).
- **Weak interaction**.
  - Interactions between the left-handed components of the fermions.
  - Mediated by **massive weak bosons** $W^\pm$ **and** Z^0 (spin one).
  - **Self interactions** between $W^\pm$ and Z^0 bosons (and photons) [see below...].
- **Strong interactions**.
  - Interactions between **colored particles** (quarks).
  - Mediated by **massless gluons** g (spin one).
  - **Self interactions** between gluons [see below...].
  - Hadrons and mesons are made of quarks and gluons.
  - At the nucleus level: binding of protons and neutrons.
- **Gravity**.
  - Interactions between all particules.
  - Mediated by the **(non-observed) massless graviton** (spin two).
  - **Not described by the Standard Model.**
  - Attempts: superstrings, M-theory, quantum loop gravity, ...

The Standard Model of particle physics - framework (1).

- **Symmetry principles ⇔ elementary particles and their interactions.**
  - * Compatible with **special relativity**.
    - ◇ **Minkowski spacetime** with the metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.
    - ◇ **Scalar product** $x\cdot y = x^\mu y_\mu = x^\mu y^\nu \eta_{\mu\nu} = x^0 y^0 - \vec{x}\cdot\vec{y}$.
    - ◇ Invariance of the **speed of light** c.
    - ◇ **Physics independent of the inertial reference frame**.
  - * Compatible with **quantum mechanics**.
    - ◇ **Classical fields**: relativistic analogous of wave functions.
  - * **Quantum field theory**.
    - ◇ **Quantization** of the fields: harmonic and fermionic oscillators.
  - * Based on **gauge theories** [see below...].

Conventions.

- * $\hbar = c = 1$ **and** $\eta_{\mu\nu}, \eta^{\mu\nu} =$ **diag**$(1, -1, -1, -1)$.
- * **Raising and lowering indices**: $V^\mu = \eta^{\mu\nu} V_\nu$ and $V_\mu = \eta_{\mu\nu} V^\nu$.
- * **Indices.**
  - ◇ Greek letters: $\mu, \nu \ldots = 0, 1, 2, 3$.
  - ◇ Roman letters: $i, j, \ldots = 1, 2, 3$.

The Standard Model of particle physics - framework (2).

- **What is a symmetry?**
  - * **A symmetry operation leaves the laws of physics invariant.**
    e.g., Newton's law is the same in any inertial frame: $\vec{F} = m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}$.
- **Examples of symmetry.**
  - * **Spacetime symmetries**: rotations, Lorentz boosts, translations.
  - * **Internal symmetries**: quantum mechanics: $|\Psi\rangle \to e^{i\alpha}|\Psi\rangle$.
- **Nœther theroem.**
  - * **To each symmetry is associated a conserved charge.**
  - * **Examples**: electric charge, energy, angular momentum, ...

The Standard Model of particle physics - framework (3).

Dynamics is based on symmetry principles.

* **Spacetime symmetries (Poincaré).**
  Particle types: scalars, spinors, vectors, ...
  Beyond: supersymmetry, extra-dimensions.
* **Internal symmetries (gauge interactions).**
  Electromagnetism, weak and strong interactions.
  Beyond: Grand Unified Theories.

- **Importance of symmetry breaking and anomalies** [see below...].
  * Masses of the **gauge bosons**.
  * Generation of the **fermion masses**.
  * **Quantum numbers** of the particles.

Outline.

Euler-Lagrange equations - theoretical concepts.

- **We consider a set of fields $\phi(x^\mu)$.**
  * They depend on **spacetime** coordinates (relativistic).
- **A system is described by a Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$** where $\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$.
  * **Variables**: the fields ϕ and their first-order derivatives $\partial_\mu \phi$.
- **Action**.
  * Related to the Lagrangian $S = \int \mathrm{d}^4 x \ \mathcal{L}$.
- **Equations of motion**.
  * Dynamics described by the **principle of least action**.
  * Leads to **Euler-Lagrange** equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} = 0 \qquad \text{where } \phi \text{ and } \partial_\mu \phi \text{ are taken independent.}$$

Euler-Lagrange equations - example.

- **The electromagnetic potential** $A^\mu(x) = \Big(V(t,\vec{x}),\ \vec{A}(t,\vec{x})\Big)$.
- **External electromagnetic current:** $j^\mu(x) = \Big(\rho(t,\vec{x}),\ \vec{\jmath}(t,\vec{x})\Big)$.
- **The system is described by the Lagrangian $\mathcal{L}$ (the action $S = \int \mathrm{d}^4x\ \mathcal{L}$).**

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

[Einstein conventions: repeated indices are summed.]

- **Equations of motion**.
  * The **Euler-Lagrange** equations are

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = 0 \rightsquigarrow \partial_\mu F^{\mu\nu} = j^\nu \rightsquigarrow \left\{ \begin{array}{lcl} \vec{\nabla}\cdot\vec{E} & = & \rho \\ \vec{\nabla}\times\vec{B} & = & \vec{\jmath} + \frac{\partial \vec{E}}{\partial t} \end{array} \right. .$$

  * The **constraint equations** come from

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \rightsquigarrow \left\{ \begin{array}{lcl} \vec{\nabla}\cdot\vec{B} & = & 0 \\ \vec{\nabla}\times\vec{E} & = & -\frac{\partial \vec{B}}{\partial t} \end{array} \right. .$$

Symmetries.

- **What is an invariant Lagrangian under a symmetry?**.
  * We associate an **operator (or matrix)** G to the symmetry:
  $$\phi(x) \to G\phi(x) \qquad \text{and} \qquad \mathcal{L} \to \mathcal{L} + \partial_\mu(\dots)\ .$$
  * The **action** is thus **invariant**.

- **Symmetries in quantum mechanics**.
  * Wigner: **G is a (anti)-unitary operator.**
  * For unitary operators, $\exists\ g$, hermitian, so that
  $$G = \exp\left[ig\right] = \exp\left[i\alpha^i g_i\right]\ .$$
    - ◇ α^i are the **transformation parameters**.
    - ◇ g_i are the symmetry **generators**.
  * **Example**: rotations $R(\vec{\alpha}) = \exp[-i\vec{\alpha}\cdot\vec{J}]$ ($\vec{J} \equiv$ angular momentum).

- **Symmetry group and algebra**.
  * The product of two symmetries is a symmetry $\Rightarrow$ $\{G_i\}$ **form a group**.
  * This implies that $\{g_i\}$ **form an algebra**.
  $$\left[g_i, g_j\right] \equiv g_i g_j - g_j g_i = i f_{ij}{}^k g_k\ .$$
  * **Rotations**: $[J_i, J_j] = iJ_k$ with (i, j, k) cyclic.

Outline.

The Poincaré group and quantum field theory.

- **Quantum mechanics** is invariant under the Galileo group.
- **Maxwell equations** are invariant under the Poincaré group.

Consistency principles.

* **Relativistic quantum mechanics.**
  Relativistic equations (Klein-Gordon, Dirac, Maxwell, ...)
* **Quantum field theory**
  The field are quantized: second quantization.
  (harmonic and fermionic oscillators).

The Poincaré algebra and the particle masses.

- **The Poincaré algebra** reads $(\mu, \nu = 0, 1, 2, 3)$

$$\begin{aligned}
\Big[L^{\mu\nu}, L^{\rho\sigma}\Big] &= -i\Big(\eta^{\nu\sigma} L^{\rho\mu} - \eta^{\mu\sigma} L^{\rho\nu} + \eta^{\nu\rho} L^{\mu\sigma} - \eta^{\mu\rho} L^{\nu\sigma}\Big) \ , \\
\Big[L^{\mu\nu}, P^{\rho}\Big] &= -i\Big(\eta^{\nu\rho} P^{\mu} - \eta^{\mu\rho} P^{\nu}\Big) \ , \\
\Big[P^{\mu}, P^{\nu}\Big] &= 0 \ ,
\end{aligned}$$

where

* $L_{\mu\nu} = -L_{\nu\mu}$ is **antisymmetric**..
* $L_{ij} = J^k \equiv$ **rotations**; (i, j, k) is a cyclic permutation of $(1, 2, 3)$.
* $L_{0i} = K^i \equiv$ **boosts** $(i = 1, 2, 3)$.
* $P_\mu \equiv$ **spacetime translations**.

Beware of the adopted conventions (especially in the literature).

- **The particle masses**.
  * A Casimir operator is an operator commuting with all generators. $\rightsquigarrow$ **quantum numbers**.
  * The quadratic Casimir $\mathcal{Q}_2$ reads $\mathcal{Q}_2 = P^\mu P_\mu = E^2 - \vec{p} \cdot \vec{p} = m^2$.
  * The masses are the **eigenvalues** of the $\mathcal{Q}_2$ operator.

Reminder: the rotation algebra and its representations.

- **The rotation algebra** reads

$$\left[J^i, J^j\right] = i\varepsilon^{ij}{}_k J^k = \begin{cases} iJ_k & \text{with } (i,j,k) \text{ a cyclic permutation of } (1,2,3). \\ -iJ_k & \text{with } (i,j,k) \text{ an anticyclic permutation of } (1,2,3). \end{cases}$$

- **The operator $\vec{J}\cdot\vec{J}$.**
  * Defining $\vec{J} = (J^1, J^2, J^3)$, we have $\left[\vec{J}\cdot\vec{J}, J^i\right] = 0$.
  * $\vec{J}\cdot\vec{J}$ is thus a **Casimir operator** (commuting with all generators).

- **Representations.**
  * A **representation** is characterized by
    - Two numbers: $j \in \frac{1}{2}\mathbb{N}$ and $m \in \{-j, -j+1, \dots, j-1, j\}$.
  * The J^i matrices are $(2j+1)\times(2j+1)$ **matrices**.
    - $j = 1/2$: Pauli matrices (over two).
    - $j = 1$: usual rotation matrices (in three dimensions).
  * A state is represented by a ket $|j, m\rangle$ such that

$$J_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle \ ,$$

$$J^3|j,m\rangle = m|j,m\rangle \qquad \text{and} \qquad \vec{J}\cdot\vec{J}|j,m\rangle = j(j+1)|j,m\rangle \ .$$

The Lorentz algebra and the particle spins.

- **The Lorentz algebra** reads

$$\Big[L^{\mu\nu}, L^{\rho\sigma}\Big] = -i\Big(\eta^{\nu\sigma}L^{\rho\mu} - \eta^{\mu\sigma}L^{\rho\nu} + \eta^{\nu\rho}L^{\mu\sigma} - \eta^{\mu\rho}L^{\nu\sigma}\Big) \ ,$$

  * We define $N^i = \frac{1}{2}(J^i + iK^i)$ and $\bar{N}^i = \frac{1}{2}(J^i - iK^i)$.

  One gets $\Big[\mathbf{N^i}, \mathbf{N^j}\Big] = -\mathbf{iN^k}$, $\Big[\mathbf{\bar{N}^i}, \mathbf{\bar{N}^j}\Big] = -\mathbf{i\bar{N}^k}$ and $\Big[\mathbf{N^i}, \mathbf{\bar{N}^j}\Big] = \mathbf{0}$.

Definition of the spin.

$$\big\{N_i\big\} \oplus \big\{\bar{N}_i\big\} = \mathfrak{sl}(2) \oplus \overline{\mathfrak{sl}(2)} \sim \mathfrak{so}(3) \oplus \mathfrak{so}(3) \ .$$

The representations of $\mathfrak{so}(3)$ are known:

$$\left\{ \begin{array}{lcl} \big\{N^i\big\} & \to & S \\ \big\{\bar{N}^i\big\} & \to & \bar{S} \end{array} \right. \quad \Rightarrow J^i = N^i + \bar{N}^i \to \text{spin} \ = S + \bar{S} \ .$$

- **The particle spins are the representations of the Lorentz algebra**.
  * $(0,0) \equiv$ **scalar** fields.
  * $(1/2, 0)$ and $(0, 1/2) \equiv$ left and right **spinors**.
  * $(1/2, 1/2) \equiv$ **vector** fields.

Representations of the Lorentz algebra (1).

- **The (four-dimensional) vector representation** $(1/2, 1/2)$.
  * Action on **four-vectors** X^μ.
  * **Generators**: a set of 10 4×4 matrices
$$(J^{\mu\nu})^\rho{}_\sigma = -i\Big(\eta^{\rho\mu}\delta^\nu{}_\sigma - \eta^{\rho\nu}\delta^\mu{}_\sigma\Big) \ .$$
  * A **finite Lorentz transformation** is given by
$$\Lambda_{(\frac{1}{2},\frac{1}{2})} = \exp\Big[\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\Big] \ ,$$
    where $\omega_{\mu\nu} \in \mathbb{R}$ are the transformation parameters.
  * **Example 1: a rotation with** $\alpha = \omega_{12} = -\omega_{21}$,
$$R(\alpha) = \exp\Big[i\alpha J^{12}\Big] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$
  * **Example 2: a boost of speed** $v = -\tanh\varphi$ **with** $\varphi = \omega_{01} = -\omega_{10}$,
$$B(\varphi) = \exp\Big[i\varphi J^{01}\Big] = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Representations of the Lorentz algebra (2).

- **Pauli matrices in four dimensions**.
  - Conventions:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

  - Definitions:

$$\sigma^\mu{}_{\alpha\dot\alpha} = (\sigma^{\mathbf{0}}, \sigma^{\mathbf{i}})_{\alpha\dot\alpha} \ , \qquad \bar\sigma^{\mu\dot\alpha\alpha} = (\sigma^{\mathbf{0}}, -\sigma^{\mathbf{i}})^{\dot\alpha\alpha}$$

  with $\alpha = 1, 2$ and $\dot\alpha = \dot 1, \dot 2$.
  The (un)dotted nature of the indices is related to Dirac spinors [see below...].

Beware of the position (lower or upper, first or second) of the indices.
Beware of the types (undotted or dotted) of the indices.

Representations of the Lorentz algebra (3).

- **The left-handed Weyl spinor representation** $(1/2, 0)$.
  - Action on **complex left-handed spinors** ψ_α $(\alpha = 1, 2)$.
  - **Generators**: a set of 10 2×2 matrices

$$(\sigma^{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}\Big(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\Big)_\alpha{}^\beta \ .$$

  - A **finite Lorentz transformation** is given by

$$\Lambda_{(\frac{1}{2},0)} = \exp\left[\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right] \ .$$

- **The right-handed Weyl spinor representation** $(0, 1/2)$.
  - Action on **complex right-handed spinors** $\bar\chi^{\dot\alpha}$ $(\dot\alpha = \dot 1, \dot 2)$.
  - **Generators**: a set of 10 2×2 matrices

$$(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = -\frac{i}{4}\Big(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\Big)^{\dot\alpha}{}_{\dot\beta} \ .$$

  - A **finite Lorentz transformation** is given by

$$\Lambda_{(0,\frac{1}{2})} = \exp\left[\frac{i}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}\right] \ .$$

- **Complex conjugation maps left-handed and right-handed spinors**.

Representations of the Lorentz algebra (4).

- **Lowering and raising spin indices**.
  * We can define a **metric acting on spin space** [Beware of the conventions],

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

$$\varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

  * One has:

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \ , \qquad \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \ , \qquad \bar\chi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta} \ , \qquad \bar\chi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta} \ .$$

Beware of the adopted conventions for the position of the indices (we are summing on the second index).

$$\varepsilon^{\alpha\beta}\psi_\beta = -\varepsilon^{\beta\alpha}\psi_\beta \ .$$

Four-component fermions: Dirac and Majorana spinors (1).

- **Dirac matrices in four dimensions (in the chiral representation)**.
  * Definition:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} .$$

  * The **(Clifford) algebra** satisfied by the γ-matrices reads

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \mathbf{2}\ \eta^{\mu\nu} .$$

  * The **chirality matrix**, *i.e.*, the fifth Dirac matrix is defined by

$$\gamma^{\mathbf{5}} = \mathbf{i}\gamma^{\mathbf{0}}\gamma^{\mathbf{1}}\gamma^{\mathbf{2}}\gamma^{\mathbf{3}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \{\gamma^{\mathbf{5}}, \gamma^\mu\} = \mathbf{0} .$$

Four-component fermions: Dirac and Majorana spinors (2).

- **A Dirac spinor is defined as**

$$\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \ ,$$

which is a **reducible** representation of the Lorentz algebra.

* **Generators of the Lorentz algebra**: a set of 10 4×4 matrices

$$\gamma^{\mu\nu} = -\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}$$

* A **finite Lorentz transformation** is given by

$$\Lambda_{(\frac12,0)\oplus(0,\frac12)} = \exp\left[\frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right] = \begin{pmatrix} \Lambda_{(\frac12,0)} & 0 \\ 0 & \Lambda_{(0,\frac12)} \end{pmatrix} \ .$$

- **A Majorana spinor is defined as**

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} \ ,$$

$\Leftrightarrow$ a Dirac spinor with **conjugate left- and right-handed components**.

$$\bar\psi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} \qquad \text{with} \qquad \bar\psi_{\dot\beta} = \Big(\psi_\beta\Big)^\dagger \ .$$

Summary - Representations and particles.

Irreducible representations of the Poincaré algebra vs. particles.

- **Scalar particles** (**Higgs boson**).
  - $(0,0)$ **representation**.
- **Massive Dirac fermions** (**quarks and leptons** after symmetry breaking).
  - $(1/2,0) \oplus (0,1/2)$ **representation**.
  - The mass term mixes both spinor representations.
- **Massive Majorana fermions** (not in the Standard Model $\Rightarrow$ **dark matter**).
  - $(1/2,0) \oplus (0,1/2)$ **representation**.
  - A Majorana field is self conjugate (the particle $=$ the antiparticle).
  - The mass term mixes both spinor representations.
- **Massless Weyl fermions** (**fermions** before symmetry breaking).
  - $(1/2,0)$ **or** $(0,1/2)$ **representation**.
  - The conjugate of a left-handed fermion is right-handed.
- **Massless and massive vector particles** (**gauge bosons**).
  - $(1/2,1/2)$ **representation**.

Outline.

Relativistic wave equations: scalar fields.

- **Definition:**
  * $(0,0)$ **representation** of the Lorentz algebra.
  * **Lorentz transformation** of a scalar field ϕ

$$\phi(x) \to \phi'(x') = \phi(x) \ .$$

- **Correspondence principle**.
  * $P_\mu \leftrightarrow i\partial_\mu$.
  * Application to the mass-energy relation: the **Klein-Gordon equation**. $P^2 = m^2 \leftrightarrow (\Box + m^2)\phi = 0$.
  * The associated Lagrangian is given by $\mathcal{L}_{\mathbf{KG}} = (\partial^\mu\phi)^\dagger(\partial_\mu\phi) - \mathbf{m^2}\phi^\dagger\phi$, cf. Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} = 0 \qquad \text{where } \phi \text{ and } \partial_\mu\phi \text{ are taken independent} \ .$$

Scalar fields in the Standard Model.

- The only undiscovered particle is a scalar field: **the Higgs boson**.
- Remark: in supersymmetry, we have a lot of scalar fields [see below...].

Relativistic wave equations: vector fields (1).

- **Definition:**
  * $(1/2, 1/2)$ **representation** of the Lorentz algebra.
  * **Lorentz transformation** of a vector field A^μ

$$A^\mu(x) \to A^{\mu\prime}(x') = \left(\Lambda_{(\frac{1}{2},\frac{1}{2})}\right)^\mu{}_\nu A^\nu(x) \ .$$

- **Maxwell equations and Lagrangian**.
  * The relativistic Maxwell equations are

$$\partial_\mu \mathbf{F}^{\mu\nu} = \mathbf{j}^\nu \ .$$

  * $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the **field strength tensor**.
  * j^ν is the **electromagnetic current**.
  * The associated Lagrangian is given by

$$\mathcal{L}_{\mathbf{EM}} = -\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} - \mathbf{A}^\mu \mathbf{j}_\mu \ .$$

  * This corresponds to an **Abelian $U(1)$ (commutative) gauge group**.

Relativistic wave equations: vector fields (2).

- **The non-abelian (non-commutative) group $SU(N)$.**
  * The group is of **dimension** $N^2 - 1$.
  * The algebra is **generated by** $N^2 - 1$ **matrices** T_a $(a = 1, \ldots, N^2 - 1)$,

$$\left[\mathbf{T_a}, \mathbf{T_b}\right] = \mathbf{i f_{ab}}^{\mathbf{c}}\ \mathbf{T_c}\ ,$$

  where $f_{ab}{}^c$ are the structure constants of the algebra.
  Example: $SU(2)$: $f_{ab}{}^c = \varepsilon_{ab}{}^c$.

  * **Usually employed representations** for model building.
    - ◇ Fundamental and anti-fundamental: $N \times N$ matrices so that

$$\mathrm{Tr}(\mathbf{T_a}) = \mathbf{0} \qquad \text{and} \qquad \mathbf{T_a^\dagger} = \mathbf{T_a}\ .$$

    - ◇ Adjoint: $(N^2 - 1) \times (N^2 - 1)$ matrices given by

$$(\mathbf{T_a})_{\mathbf{b}}{}^{\mathbf{c}} = -\mathbf{i f_{ab}}^{\mathbf{c}}\ .$$

  * For a given representation $\mathcal{R}$:

$$\mathrm{Tr}\big(\mathbf{T_a}\mathbf{T_b}\big) = \tau_{\mathcal{R}}\delta_{\mathbf{ab}}\ ,$$

  where $\tau_{\mathcal{R}}$ is the **Dynkin index** of the representation.

Relativistic wave equations: vector fields (3).

- **Application to physics**.
  - * We select a **gauge group** (here: $SU(N)$).
  - * We define a **coupling constant** (here g).
  - * We assign **representations** of the group to matter fields.
  - * **The $\mathbf{N^2-1}$ gauge bosons** are given by $\mathbf{A}^\mu = \mathbf{A}^{\mu a}\mathbf{T}_a$.
  - * The **field strength tensor** is defined by

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - \mathbf{ig}[\mathbf{A}_\mu, \mathbf{A}_\nu] \\ &= \left[\partial_\mu \mathbf{A}^c_\nu - \partial_\nu \mathbf{A}^c_\mu + \mathbf{g}\ \mathbf{f}_{ab}{}^c \mathbf{A}^a_\mu \mathbf{A}^b_\nu\right]\ \mathbf{T}_c\ . \end{aligned}$$

  - * The associated Lagrangian is given by [Yang, Mills (1954)]

$$\mathcal{L}_{\mathbf{YM}} = -\frac{\mathbf{1}}{\mathbf{4}\tau_{\mathcal{R}}} \mathbf{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu})\ .$$

  - * Contains **self interactions** of the vector fields.

Vector fields in the Standard Model.

- The gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$ [see below...].
- The bosons are **the photon, the weak $W^\pm$ and Z^0 bosons, and the gluons**.

Relativistic wave equations: Dirac spinors.

- **Definition:**
  - * $(1/2, 0) \oplus (0, 1/2)$ **representation** of the Poincaré algebra.
  - * **Lorentz transformations** of a Dirac field ψ_D

$$\psi_D(x) \to \psi'_D(x') = \Lambda_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}\psi_D(x) \ .$$

- **Dirac's idea.**
  - * The Klein-Gordon equation is quadratic $\Rightarrow$ particles and antiparticles.
  - * **A conceptual problem in the 1920's**.
  - * Linearization of the d'Alembertian:

$$(\mathbf{i}\gamma^\mu\partial_\mu - \mathbf{m})\psi_\mathbf{D} = \mathbf{0} \qquad \Leftrightarrow \qquad \mathcal{L}_\mathbf{D} = \bar{\psi}_\mathbf{D}(\mathbf{i}\gamma^\mu\partial_\mu - \mathbf{m})\psi_\mathbf{D} \ ,$$

where
  - $\bar{\psi}_D = \psi_D^\dagger\gamma^0$.
  - $(\gamma^\mu\partial_\mu)^2 = \Box \Leftrightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Fermionic fields in the Standard Model.

- Matter $\equiv$ **Dirac spinors after symmetry breaking**.
- Matter $\equiv$ **Weyl spinors before symmetry breaking** [see below...].

Summary - Relativistic wave equations.

Relativistic wave equations.

- **General properties**.
  - * The equations derive from **Poincaré invariance**.
- **Scalar particles** (**Higgs boson**).
  - * **Klein-Gordon equation**.
- **Massive Dirac and Majorana fermions** (**quarks and leptons**).
  - * **Dirac equation**.
- **Massless and massive vector particles** (**gauge bosons**).
  - * **Maxwell equations** (Abelian case).
  - * **Yang-Mills equations** (non-Abelian case).

Outline.

Global symmetries for the Dirac Lagrangian.

- **Toy model.**
  - * We select the **gauge group** $SU(N)$ with a coupling constant g.
  - * We assign the **fundamental representations** to the fermion fields Ψ,

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} , \qquad \bar{\Psi} = \begin{pmatrix} \bar{\psi}_1 & \cdots & \bar{\psi}_N \end{pmatrix} .$$

  - * The **Lagrangian** reads

$$\mathcal{L} = \bar{\Psi}\Big(i\gamma^\mu \partial_\mu - m\Big)\Psi .$$

- **The global $SU(N)$ invariance.**
  - * We define a **global $SU(N)$ transformation** of parameters ω^a,

$$\Psi(x) \to \Psi'(x) = \exp\Big[+ ig\omega^a T_a^{\rm fund}\Big]\Psi \equiv U\ \Psi ,$$
$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}\exp\Big[- ig\omega^a T_a^{\rm fund}\Big] \equiv \bar{\Psi}\ U^\dagger .$$

  - * **The Lagrangian is invariant**,

$$\mathcal{L} \to \mathcal{L} .$$

Gauge symmetries for the Dirac Lagrangian (1).

- **Local (internal) $SU(N)$ invariance.**
  - * **Promotion** of the global invariance to a local invariance.
  - * We define a **local $SU(N)$ transformation** of parameters $\omega^a(x)$,

$$\Psi(x) \to \Psi'(x) = U(x)\ \Psi\ , \qquad \bar\Psi(x) \to \bar\Psi'(x) = \bar\Psi\ U^\dagger(x)\ .$$

  - * **The Lagrangian is not invariant anymore**.

$$\begin{aligned} \partial_\mu \Psi(x) &\quad \not\longrightarrow \quad U(x)\ \partial_\mu \Psi(x) \\ \mathcal{L} = \bar\Psi\Big(i\gamma^\mu\partial_\mu - m\Big)\Psi &\quad \not\longrightarrow \quad \mathcal{L}\ . \end{aligned}$$

  - * **Due to**:
    - ◇ The spacetime dependence of $U(x)$.
    - ◇ The presence of derivatives in the Lagrangian.
  - * **Idea**: modification of the derivative.
    - ◇ Introduction of a new field with *ad hoc* transformation rules.
    - ◇ Recovery of the Lagrangian invariance.

Gauge symmetries for the Dirac Lagrangian (2).

- **Local (internal) $SU(N)$ invariance.**
  - * Local invariance is recovered after:
    - ◇ The introduction of a **new vector field** $A^\mu = A^{\mu a} T_a^{\rm fund}$ with

$$A^\mu(x) \to A^{\mu\prime}(x) = U(x)\Big[A^\mu(x) + \frac{i}{g}\partial^\mu\Big]U^\dagger(x) \ ,$$

$$F^{\mu\nu}(x) \to U(x)F^{\mu\nu}(x)U^\dagger(x) \quad \Rightarrow \quad \mathrm{Tr}\big(F^{\mu\nu}F_{\mu\nu}\big) \to \mathrm{Tr}\big(F^{\mu\nu}F_{\mu\nu}\big) \ .$$

    - ◇ The modification of the derivative into a **covariant derivative**,

$$\partial_\mu\Psi(x) \to D_\mu\Psi(x) = \Big[\partial_\mu - igA_\mu(x)\Big]\Psi(x) \ .$$

    - ◇ **Transformation laws**:

$$D_\mu\Psi(x) \ \to \ U(x)\ D_\mu\Psi(x) \quad \Rightarrow \quad \mathcal{L} \ \to \ \mathcal{L} \ .$$

- **This holds (and simplifies) for $U(1)$ gauge invariance.** In particular:

$$A^\mu(x) \to A^{\mu\prime}(x) = A^\mu(x) + \partial^\mu\omega(x) \ .$$

- **Example: Abelian $U(1)_{e.m.}$ gauge group for electromagnetism**.

Symmetry breaking - theoretical setup.

- **Let us consider a $U(1)_X$ gauge symmetry.**
  * Gauge boson $\mathbf{X}_\mu$ − gauge coupling constant $\mathbf{g}_X$.
- **Matter content.**
  * A set of fermionic particles $\boldsymbol{\Psi}^j$ of charge $\mathbf{q}^j_X$.
  * A complex scalar field ϕ with charge $\mathbf{q}_\phi$.
- **Lagrangian.**
  * **Kinetic and gauge interaction terms for all fields.**

$$\begin{aligned}\mathcal{L}_{\rm kin} =& -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\Psi}_j\ i\gamma^\mu D_\mu\ \Psi^j + (D_\mu\phi)^\dagger (D^\mu\phi)\\ =& -\frac{1}{4}\Big(\partial_\mu X_\nu - \partial_\nu X_\mu\Big)\Big(\partial^\mu X^\nu - \partial^\nu X^\mu\Big)\\ & + \bar{\Psi}_j\gamma^\mu\Big(i\partial_\mu + g_X q^j_X X_\mu\Big)\Psi^j + \Big[\big(\partial_\mu + i g_X q_\phi X_\mu\big)\phi^\dagger\Big]\Big[\big(\partial^\mu - i g_X q_\phi X^\mu\big)\phi\Big]\ .\end{aligned}$$

  * **A scalar potential ($\mathcal{L}_V = -V_{\rm scal}$) and Yukawa interactions.**

$$V_{\rm scal} = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \qquad \text{with} \quad \lambda > 0\ , \quad \mu^2 > 0\ ,$$

$$\mathcal{L}_{\rm Yuk} = -y_j\phi\bar{\Psi}_j\Psi^j + \text{h.c.} \qquad \text{with } \mathbf{y_j} \text{ being the Yukawa coupling.}$$

Symmetry breaking - minimization of the scalar potential.

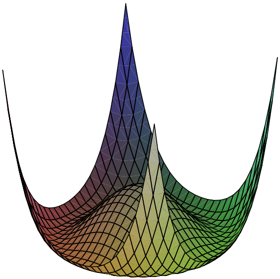
- The system lies at the **minimum** of the potential.

$$\frac{\mathrm{d}V_{\mathrm{scal}}}{\mathrm{d}\phi} = 0 \Leftrightarrow \langle\phi\rangle = \frac{1}{\sqrt{2}}\sqrt{\frac{\mu^2}{\lambda}}e^{i\alpha_0}\ .$$

- $\mathbf{v} = \sqrt{\mathbf{2}}\langle\phi\rangle$ is the vacuum expectation value (**vev**) of the field ϕ.
- We define ϕ such that $\alpha_0 = 0$.
- We **shift** the scalar field by its vev

$$\phi = \frac{1}{\sqrt{2}}\Big[v + A + i\ B\Big]\ ,$$

where A and B are **real** scalar fields.



$$V_{\mathrm{scal}} = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\ .$$

Symmetry breaking - mass eigenstates (1).

We shift the scalar field by its vev.

$$\phi = \frac{1}{\sqrt{2}}\Big[\mathbf{v} + \mathbf{A} + \mathbf{i}\ \mathbf{B}\Big]\ .$$

- **Scalar mass eigenstates.**
  - The scalar potentiel reads now

$$V_{\rm scal} = \lambda v^2 A^2 + \lambda\Big[\frac{1}{4}A^4 + \frac{1}{4}B^4 + \frac{1}{2}A^2B^2 + vA^3 + vAB^2\Big]\ .$$

  - One gets **self interactions** between A and B.
  - A is a **massive** real scalar field, $\mathbf{m_A^2 = 2\mu^2}$, the so-called **Higgs boson**.
  - B is a **massless** pseudoscalar field, the so-called **Goldstone boson**.

Symmetry breaking - mass eigenstates (2).

We shift the scalar field by its vev.

$$\phi = \frac{1}{\sqrt{2}}\Big[\mathbf{v} + \mathbf{A} + \mathbf{i}\ \mathbf{B}\Big] \ .$$

- **Gauge boson mass m_X.**
  * The kinetic and gauge interaction terms for the scalar field ϕ read now

$$\Big(D^\mu\phi^\dagger\Big)\Big(D_\mu\phi\Big) = \Big[\big(\partial_\mu + i g_X q_\phi X_\mu\big)\phi^\dagger\Big]\Big[\big(\partial^\mu - i g_X q_\phi X^\mu\big)\phi\Big]$$
$$= \frac{1}{2}\partial_\mu \mathbf{A}\partial^\mu \mathbf{A} + \frac{1}{2}\partial_\mu \mathbf{B}\partial^\mu \mathbf{B} + \frac{1}{2}\mathbf{g}_\mathbf{X}^2\mathbf{v}^2\mathbf{X}_\mu\mathbf{X}^\mu + \ldots$$

  * One gets **kinetic terms** for the A and B fields.
  * The dots stand for **bilinear and trilinear interactions** of A, B and X_μ.
  * The gauge boson becomes massive, $\mathbf{m_X = g_X v}$.
  * The Goldstone boson is eaten $\equiv$ the third polarization state of X_μ.
  * **The gauge symmetry is spontaneously broken.**

Symmetry breaking - mass eigenstates (3).

We shift the scalar field by its vev.

$$\phi = \frac{1}{\sqrt{2}}\Big[\mathbf{v} + \mathbf{A} + \mathbf{i}\ \mathbf{B}\Big] \ .$$

- **Fermion masses m_j.**
  - The Yukawa interactions read now

$$\mathcal{L}_{\rm Yuk} = -y_j \phi \bar\Psi_j \Psi^j \to \frac{1}{\sqrt{2}}\ \mathbf{y_j v}\ \bar{\mathbf{\Psi}}_\mathbf{j} \mathbf{\Psi}^\mathbf{j} + \frac{1}{\sqrt{2}}\ \mathbf{y_j}\ (\mathbf{A} + \mathbf{i}\ \mathbf{B})\ \bar{\mathbf{\Psi}}_\mathbf{j} \mathbf{\Psi}^\mathbf{j} \ .$$

  - One gets **Yukawa interactions** between A, B and Ψ^j.
  - The fermion fields become massive, $\mathbf{m_j} = \mathbf{y_j v}$.

Summary - Nœther procedure.

Nœther procedure to get gauge invariant Lagrangians.

1. Choose a **gauge group**.
2. Setup the **matter field content** in a given representation.
3. Start from the **free Lagrangian for matter fields**.
4. Promote derivatives to **covariant derivatives**.
5. Add **kinetic terms** for the gauge bosons ($\mathcal{L}_{YM}$ or $\mathcal{L}_M$).

- **Some remarks:**
  * The Nœther procedure holds for both **fermion and scalar** fields.
  * This implies that the **interactions are dictated by the geometry**.
  * The gauge group and matter content are **not predicted**.
  * The symmetry can be eventually **broken**.
  * The theory **must be anomaly-free**.
  * This holds in **any number of spacetime dimensions**.
  * This can be generalized to **superfields** (supersymmetry, supergravity).

Outline.

Theoretical setup.

- **The electromagnetism is the simplest gauge theory**.
- **We consider an Abelian gauge group, $U(1)_{e.m.}$.**
  - * Gauge boson: the photon $\mathbf{A}_\mu$.
  - * Gauge coupling constant: the electromagnetic coupling constant **e**.
  - We relate e to $\alpha = \frac{e^2}{4\pi}$.
  - **Both quantities depend on the energy** (cf. renormalization):

$$\alpha(0) \approx \frac{1}{137} \qquad \text{and} \qquad \alpha(100\text{GeV}) \approx \frac{1}{128} \ .$$

- **Matter content.**

Name	Field: $1^{\rm st}$ gen.	$2^{\rm nd}$ gen.	$3^{\rm rd}$ gen.	Electric charge q
Charged lepton	Ψ_e	Ψ_μ	Ψ_τ	-1
Neutrino	Ψ_{ν_e}	Ψ_{ν_μ}	Ψ_{ν_τ}	0
Up-type quarks	Ψ_u	Ψ_c	Ψ_t	$2/3$
Down-type quarks	Ψ_d	Ψ_s	Ψ_b	$-1/3$

Lagrangian.

- **We start from the free Lagrangian**,

$$\mathcal{L}_{\rm free} = \sum_{j=e,\nu_e,u,d,\ldots} \bar\Psi_j \; i\gamma^\mu \partial_\mu \Psi^j \ .$$

- **The Nœther procedure leads to**

$$\mathcal{L}_{\rm QED} = \sum_{j=e,\nu_e,u,d,\ldots} \bar\Psi_j \; i\gamma^\mu D_\mu \Psi^j - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \left\{ \begin{array}{l} D_\mu = \partial_\mu - ieqA_\mu \ , \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \end{array} \right.$$

- **The electromagnetic interactions are given by**

$$\mathcal{L}_{\rm int} = \sum_{j=e,u,d,\ldots} \bar\Psi_j \; eq\gamma^\mu A_\mu \Psi^j \ .$$

$\equiv$ **photon-fermion-antifermion vertices**:

* $\gamma^\mu \rightsquigarrow$ the fermions couple through their **spin**.
* **q** $\rightsquigarrow$ the fermions couple through their **electric charge**.

Outline.

From Lagrangians to practical computations (1).

- **Scattering theory**.
  - * Initial state $i(t)$ at a date t.
  - * Evolution to a date t'
  - * Transition to a final state $f(t')$ (at the date t').
  - * **The transition is related to the so-called S-matrix**:

$$S_{fi} = \Big\langle f(t') \ \Big| \ i(t') \Big\rangle = \Big\langle f(t') \ \Big| \ S \ \Big| \ i(t) \Big\rangle \ .$$

- **Perturbative calculation of S_{fi}**.
  - * S_{fi} is related to the **path integral**

$$\int \mathrm{d}\big(\text{fields}\big) \ e^{i \int \mathrm{d}^4 x \mathcal{L}(x)} \ ,$$

  - * S_{fi} can be **perturbatively expanded as**:

$$\begin{aligned} S_{fi} &= \delta_{fi} + i \Big[\int \mathrm{d}^4 x \mathcal{L}(x) \Big]_{fi} - \frac{1}{2} \Big[\int \mathrm{d}^4 x \mathrm{d}^4 x' \, T\Big\{ \mathcal{L}(x) \mathcal{L}(x') \Big\} \Big]_{fi} + \ldots \\ &= \text{no interaction} \ + \ \text{one interaction} \ + \ \text{two interactions} \ + \ldots \\ &= \delta_{fi} + i T_{fi} \ . \end{aligned}$$

- **We need to calculate T_{fi}**.

From Lagrangians to practical computations (2).

- **Example in QED with one interaction: the $e^+e^- \to \gamma$ process**.
  - * **The Lagrangian** is given by

$$\mathcal{L}_{\rm QED} = \sum_{j=e,\nu_e,u,d,\ldots} \bar\Psi_j \, i\gamma^\mu D_\mu \Psi^j - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \left\{ \begin{array}{l} D_\mu = \partial_\mu - ieqA_\mu \ , \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \end{array} \right.$$

  - * **Initial state $i = e^+e^-$** and **final state: $f = \gamma$**.
  - * One single **interaction term** containing the Ψ_e, $\bar\Psi_e$ and A_μ fields.

$$\mathcal{L}_{\rm QED} \Rightarrow -e \, \bar\Psi_e \, \gamma^\mu A_\mu \Psi^e \ .$$

  - * The corresponding contribution to S_{fi} reads

$$i \Big[\int \mathrm{d}^4 x \mathcal{L}(x) \Big]_{\mathbf{fi}} = i \int \mathrm{d}^4 x \, \Big[- e \, \bar\Psi_e \, \gamma^\mu A_\mu \Psi^e \Big] \ .$$

- **More than one interaction**.
  - * **Intermediate, virtual particles are allowed**.
    e.g.: $e^+e^- \to \mu^+\mu^- \rightsquigarrow e^+e^- \to \gamma^* \to \mu^+\mu^-$.
  - * **Same principles**, but accounting in addition for chronology.

From Lagrangians to practical computations (3).

- **We consider the specific process $i_1(p_a) + i_2(p_b) \to f_1(p_1) + \ldots + f_n(p_n)$.**
  * **The initial state** is $i(t) = i_1(p_a), i_2(p_b)$ (as in colliders).
  * **The *n*-particle final state** is $f(t') = f_1(p_1), \ldots, f_n(p_n)$.
  * p_a, p_b, p_1,..., and p_n are the four-momenta.
- **We solve the equations of motion and the fields are expanded as plane waves**.

$$\psi = \int \mathrm{d}^4 p \left[(\ldots)e^{-ip\cdot x} + (\ldots)e^{+ip\cdot x}\right] \ \ldots$$

  * The unspecified terms correspond to **annihilation/creation operators of (anti)particles** (harmonic and fermionic oscillators).

- **We inject these solutions in the Lagrangian**.
  * Integrating the exponentials leads to **momentum conservation**.

$$\int \mathrm{d}^4 x \left[e^{-ip_a\cdot x} e^{-ip_b\cdot x} \prod_j e^{-ip_j\cdot x}\right] = (2\pi)^4 \ \delta^{(4)}\Big(p_a + p_b - \sum_j p_j\Big) \ .$$

From Lagrangians to practical computations (4).

- **We define the matrix element**.

$$iT_{fi} = (2\pi)^4\ \delta^{(4)}\Big(p_a + p_b - \sum_j p_j\Big) iM_{fi}\ .$$

- **By definition, the total cross section**:
  - * Is the **total production rate** of the final state from the initial state.
  - * Requires an **integration over all final state configurations**.
  - * Requires an **average over all initial state configurations**.

$$\sigma = \frac{1}{F} \int \mathrm{dPS}^{(n)} \overline{\left|M_{fi}\right|^2}\ .$$

- **The differential cross section with respect to a kinematical variable** ω is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = \frac{1}{F} \int \mathrm{dPS}^{(n)} \overline{\left|M_{fi}\right|^2} \delta\Big(\omega - \omega(p_a, p_b, p_1, \dots, p_n)\Big)\ .$$

From Lagrangians to practical computations (5).

Total cross section.

$$\sigma = \frac{1}{F}\int \mathrm{dPS}^{(n)}\,\overline{|M_{fi}|^2}\ .$$

- **The integration over phase space (*cf.* final state) reads**

$$\int \mathrm{dPS}^{(n)} = \int (2\pi)^4\ \delta^{(4)}\Big(p_a + p_b - \sum_j p_j\Big) \prod_j \left[\frac{\mathrm{d}^4 p_j}{(2\pi)^4}(2\pi)\delta(p_j^2 - m_j^2)\theta(p_j^0)\right]\ .$$

  * It includes **momentum conservation**.
  * It includes **mass-shell conditions**.
  * The energy is **positive**.
  * We integrate over all final state momentum configurations.

- **The flux factor F (*cf.* initial state)** reads

$$\frac{1}{F} = \frac{1}{4\sqrt{(p_a\cdot p_b)^2 - m_a^2 m_b^2}}\ .$$

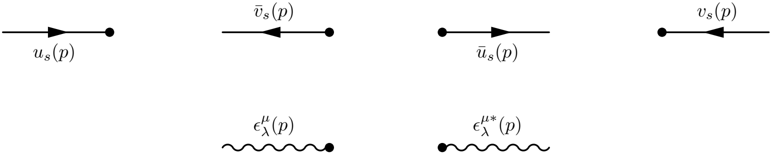
  * It normalizes σ with respect to the initial state density by surface unit.

From Lagrangians to practical computations (6).

- **The squared matrix element $\overline{|M_{fi}|^2}$**
  * Is **averaged** over the initial state quantum numbers and spins.
  * Is **summed** over the final state quantum numbers and spins.
  * Can be calculated with the **Feynman rules** derived from the Lagrangian.
    - ◇ **External particles**: spinors, polarization vectors, ...
    - ◇ **Intermediate particles**: propagators.
    - ◇ **Interaction vertices**.
- **External particles**.
  * Rules derived from the **solutions of the equations of motion**.
- **Propagators**.
  * Rules derived from the **free Lagrangians**.
- **Vertices**.
  * Rules directly extracted from the **interaction terms** of the Lagrangian.

From Lagrangians to practical computations (7).

- **Feynman rules for external particles (spinors, polarization vectors).**



* Obtained after **solving Dirac and Maxwell equations**.

$$\psi = \int \mathrm{d}^4 p \left[(\dots) e^{-ip\cdot x} + (\dots) e^{+ip\cdot x} \right] \ \dots$$

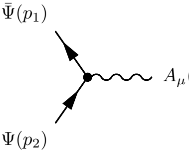
* They are **the physical degrees of freedom** (included in the dots).
* We **do not need** their explicit forms for practical calculations [see below...].

From Lagrangians to practical computations (8).

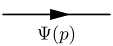
- **Interactions and propagators.**

QED Lagrangian.

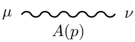
$$\mathcal{L}_{\rm QED} = \sum_{\mathbf{j=e,\nu_e,u,d,\ldots}} \bar{\mathbf{\Psi}}_\mathbf{j}\ \mathbf{i}\gamma^\mu\Big(\partial_\mu - \mathbf{ieqA}_\mu\Big)\mathbf{\Psi}^\mathbf{j} - \frac{\mathbf{1}}{\mathbf{4}}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\ .$$



$\mathbf{ieq}\gamma^\mu$



$$\mathbf{i}\frac{\gamma^\mu \mathbf{p}_\mu + \mathbf{m}}{\mathbf{p}^2 - \mathbf{m}^2}$$

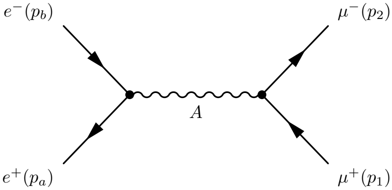


$$\frac{-\mathbf{i}\eta^{\mu\nu}}{\mathbf{p}^2}$$

- **We need to fix the gauge to derive the photon propagator.**
  $\rightsquigarrow$ **Feynman gauge: $\partial_\mu A^\mu = 0$.**
- **Any other theory would lead to similar rules.**

Example of a calculation: $e^+e^- \to \mu^+\mu^-$ (1).

- **Drawing of the Feynman diagram, using the available Feynman rules**.



- **Amplitude *iM* from the Feynman rules (following reversely the fermion lines).**

$$iM = \Big[\bar{v}_{s_a}(p_a)\ (-ie\gamma^\mu)\ u_{s_b}(p_b)\Big]\Big[\bar{u}_{s_2}(p_2)\ (-ie\gamma^\nu)\ v_{s_1}(p_1)\Big]\frac{-i\eta_{\mu\nu}}{(p_a+p_b)^2}\ .$$

Example of a calculation: $e^+e^- \to \mu^+\mu^-$ (2).

- **Derivation of the conjugate amplitude $-iM^\dagger$.**

$$iM = \Big[\bar{v}_{s_a}(p_a)\ (-ie\gamma^\mu)\ u_{s_b}(p_b)\Big]\Big[\bar{u}_{s_2}(p_2)\ (-ie\gamma^\nu)\ v_{s_1}(p_1)\Big]\frac{-i\eta_{\mu\nu}}{(p_a+p_b)^2}\ ,$$

$$-iM^\dagger = \Big[\bar{u}_{s_b}(p_b)\ (ie\gamma^\mu)\ v_{s_a}(p_a)\Big]\Big[\bar{v}_{s_1}(p_1)\ (ie\gamma^\nu)\ u_{s_2}(p_2)\Big]\frac{i\eta_{\mu\nu}}{(p_a+p_b)^2}\ .$$

  * Definitions: $\bar{u} = u^\dagger\gamma^0$ and $\bar{v} = v^\dagger\gamma^0$.
  * We remind that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.
  * We remind that $\gamma^0\gamma^0 = 1$ and $(\gamma^0)^\dagger = \gamma^0$.

- **Computation of the squared matrix element $\overline{|M|^2}$.**

$$\overline{|M|^2} = \frac{1}{2}\ \frac{1}{2}\ (iM)\ \big(-iM^\dagger\big)\ .$$

  * We average over the **initial electron spin** $\rightsquigarrow 1/2$.
  * We average over the **initial positron spin** $\rightsquigarrow 1/2$.

Example of a calculation: $e^+e^- \to \mu^+\mu^-$ (3).

- **Computation of the squared matrix element $\overline{|M|^2}$.**

$$\overline{|M|^2} = \frac{e^4}{4(p_a+p_b)^4} \mathrm{Tr}\Big[\gamma^\mu(\not{p}_b + m_e)\gamma^\rho(\not{p}_a - m_e)\Big] \, \mathrm{Tr}\Big[\gamma_\mu(\not{p}_1 - m_\mu)\gamma_\rho(\not{p}_2 + m_\mu)\Big] \ .$$

  - * We have performed a **sum over all the particle spins**.
  - * We have introduced $\not{p} = \gamma^\nu \mathbf{p}_\nu$, the electron and muon masses $\mathbf{m_e}$ and $\mathbf{m}_\mu$.
  - * We have used the properties derived from the Dirac equation

$$\sum_{\mathbf{s}} \mathbf{u_s(p)\bar{u}_s(p)} = \not{p} + \mathbf{m} \quad \text{and} \quad \sum_{\mathbf{s}} \mathbf{v_s(p)\bar{v}_s(p)} = \not{p} - \mathbf{m} \ .$$

  - * For completeness, Maxwell equations tell us that

$$\sum_\lambda \epsilon^\mu_\lambda(\mathbf{p})\epsilon^{\nu\,*}_\lambda(\mathbf{p}) = -\eta^{\mu\nu} \ .$$

[This relation is gauge-dependent.]

Example of a calculation: $e^+e^- \to \mu^+\mu^-$ (4).

- **Simplification of the traces, in the massless case.**

$$\overline{|M|^2} = \frac{8e^4}{(p_a+p_b)^4}\Big[(p_b\cdot p_1)(p_a\cdot p_2) + (p_b\cdot p_2)(p_a\cdot p_1)\Big]\ .$$

* We have used the properties of the Dirac matrices

$$\begin{aligned}
\mathrm{Tr}\Big[\gamma^{\mu_1}\ldots\gamma^{\mu_{2k+1}}\Big] &= 0\ ,\\
\mathrm{Tr}\Big[\gamma^{\mu}\gamma^{\nu}\Big] &= 4\eta^{\mu\nu}\ ,\\
\mathrm{Tr}\Big[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\Big] &= 4\Big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\Big)\ ,\\
\mathrm{Tr}\Big[\gamma^{5}\Big] &= 0\ ,\\
\mathrm{Tr}\Big[\gamma^{5}\gamma^{\mu}\gamma^{\nu}\Big] &= 0\ ,\\
\mathrm{Tr}\Big[\gamma^{5}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\Big] &= 4i\epsilon^{\mu\nu\rho\sigma} \quad \text{with} \quad \epsilon_{0123} = 1\ .
\end{aligned}$$

Example of a calculation: $e^+e^- \to \mu^+\mu^-$ (5).

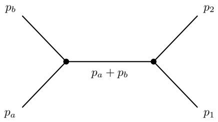
- **Mandelstam variables and differential cross section.**

$$\overline{|M|^2} = \frac{2e^4}{s^2}\left[t^2+u^2\right] \Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{e^4}{8\pi s^4}\left[t^2+u^2\right] \ .$$

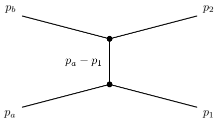
  * We have introduced the Mandelstam variables

$$\begin{aligned} \mathbf{s} &= (\mathbf{p_a}+\mathbf{p_b})^2 = (\mathbf{p_1}+\mathbf{p_2})^2 \ , \\ \mathbf{t} &= (\mathbf{p_a}-\mathbf{p_1})^2 = (\mathbf{p_b}-\mathbf{p_2})^2 \ , \\ \mathbf{u} &= (\mathbf{p_a}-\mathbf{p_2})^2 = (\mathbf{p_b}-\mathbf{p_1})^2 \ . \end{aligned}$$

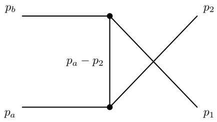
- **Remark: sub-processes names according to the propagator.**



***s*-channel**



***t*-channel**



***u*-channel**

Summary - Matrix elements from Feynman rules.

Calculation of a matrix element.

1. Extraction of the **Feynman rules** from the Lagrangian.
2. Drawing of all possible **Feynman diagrams** for the considered process.
3. Derivation of the **transition amplitudes** using the Feyman rules.
4. Calculation of the squared **matrix element**.
   * Sum/average over **final/initial internal quantum numbers**.
   * Calculation of **traces of Dirac matrices**.
   * Possible use of the **Mandelstam variables**.

Outline.

The Fermi model of weak interactions (1).

- **Proton decay (Hahn and Meitner, 1911).**

$$p \longrightarrow n + e^+ \ .$$

  - Momentum conservation **fixes final state energies to a single value** (depending on the proton energy).
  - Observation: **the energy spectrum of the electron is continuous**.

- **Solution (Pauli, 1930): introduction of the neutrino**.

$$p \longrightarrow n + e^+ + \nu_e \ \Leftrightarrow \mathbf{u} \longrightarrow \mathbf{d} + \mathbf{e}^+ + \nu_{\mathbf{e}} \qquad \text{at the quark level.}$$

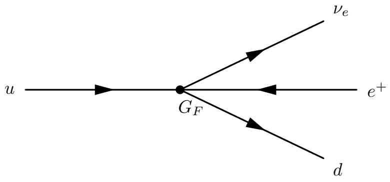
  - Reminder: $p = uud$ (naively).
  - Reminder: $n = udd$ (naively).
  - $1 \longrightarrow 3$ particle process: **continuous electron energy spectrum**.

- **How to construct a Lagrangian describing beta decays?**.

The Fermi model of weak interactions (2).

- **Phenomenological model based on four-point interactions (Fermi, 1932)**.

$$\mathcal{L}_{\rm Fermi} = -2\sqrt{2} G_F \Big[\bar\Psi_d \ \gamma_\mu \frac{1-\gamma^5}{2} \ \Psi_u\Big]\Big[\bar\Psi_{\nu_e} \ \gamma^\mu \frac{1-\gamma^5}{2} \ \Psi_e\Big] + \text{h.c.} \ .$$



* **Phenomenological model** $\Leftrightarrow$ reproducing experimental data.
* Based on **four-fermion interactions**.
* The coupling constant G_F is **measured**.
* $G_F = 1.163710^{-5}$ GeV^{-2} is **dimensionful**.

The Fermi model of weak interactions (3).

- **The Fermi Lagrangian can be rewritten as**

$$\mathcal{L}_{\rm Fermi} = -2\sqrt{2}G_F\Big[\bar\Psi_d\ \gamma_\mu \frac{1-\gamma^5}{2}\ \Psi_u\Big]\Big[\bar\Psi_{\nu_e}\ \gamma^\mu \frac{1-\gamma^5}{2}\ \Psi_e\Big] + \text{h.c.}$$
$$= -2\sqrt{2}G_F \mathbf{H}_\mu \mathbf{L}^\mu + \text{h.c.}\ .$$

  * It contains a **leptonic piece** L^μ and a **quark piece** H_μ.
  * Both pieces have the **same structure**.

- **The structure of the weak interactions**
  * The leptonic piece L^μ has a $V - A$ structure:

$$L^\mu = \bar\Psi_{\nu_e}\gamma^\mu\frac{1-\gamma^5}{2}\Psi_e = \frac{1}{2}\bar\Psi_{\nu_e}\gamma^\mu\Psi_e - \frac{1}{2}\bar\Psi_{\nu_e}\gamma^\mu\gamma^5\Psi_e\ .$$

  * Similarly, the quark piece H_μ has a $V - A$ structure.
  * **The Fermi Lagrangian contains thus VV, AA and VA terms**.

- **Behavior under parity transformations**.
  * Under a parity transformation: $V \to -V$ and $A \to A$.
  * The **VA terms (and thus weak interactions) violate parity**.
  * Parity violation has been observed experimentally (Wu *et al.*, 1956).

The Fermi model of weak interactions (4).

- **Analysis of the currents L^μ and H^μ.**

$$L^\mu = \bar{\Psi}_{\nu_e}\gamma^\mu \frac{1-\gamma^5}{2}\Psi_e \qquad \text{and} \qquad H^\mu = \bar{\Psi}_d\gamma^\mu \frac{1-\gamma^5}{2}\Psi_u \ .$$

  * Presence of **the left-handed chirality projector** $P_L = (1-\gamma^5)/2$.

- **Projectors and their properties.**
  * The **chirality projectors** are given by

$$P_L = \frac{1-\gamma^5}{2} \qquad \text{and} \qquad P_R = \frac{1+\gamma^5}{2} \ .$$

  * They fulfill the **properties**

$$P_L + P_R = 1 \ , \qquad P_L^2 = P_L \qquad \text{and} \qquad P_R^2 = P_R \ .$$

  * If Ψ is a Dirac spinor, **left and right** associated spinors are recovered by

$$\Psi_D = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \ , \qquad \Psi_L = P_L\Psi_D = \begin{pmatrix}\psi_L\\ 0\end{pmatrix} \ , \qquad \Psi_R = P_R\Psi_D = \begin{pmatrix}0\\ \psi_R\end{pmatrix} \ .$$

- **Only left-handed fermions are sensitive to the weak interactions.**

The Fermi model of weak interactions (5).

- **Introducting the left-handed chirality projector $P_L = 1/2(1-\gamma^5)$**:

$$L^\mu = \bar\Psi_{\nu_e}\gamma^\mu P_L \Psi_e = \bar\Psi_{\nu_e,L}\gamma^\mu\Psi_{e,L} \quad \text{and} \quad (L^\mu)^\dagger = \bar\Psi_e\gamma^\mu P_L\Psi_{\nu_e} = \bar\Psi_{e,L}\gamma^\mu\Psi_{\nu_e,L} \ ,$$
$$H^\mu = \bar\Psi_d\gamma^\mu P_L\Psi_u = \bar\Psi_{d,L}\gamma^\mu\Psi_{u,L} \quad \text{and} \quad (H^\mu)^\dagger = \bar\Psi_u\gamma^\mu P_L\Psi_d = \bar\Psi_{u,L}\gamma^\mu\Psi_{d,L} \ .$$

- **Behavior of the fields under the weak interacions**.
  - * **Left-handed electron and neutrino** behave similarly.
  - * **Up and down quarks** behave similarly.

- **Idea: group into doublets the left-handed components of the fields**:

$$L_e = \begin{pmatrix}\Psi_{\nu_e,L}\\ \Psi_{e,L}\end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix}\Psi_{u,L}\\ \Psi_{d,L}\end{pmatrix} \ .$$

- **The currents are then rewritten as**:

$$L^\mu = \bar\Psi_{\nu_e,L}\gamma^\mu\Psi_{e,L} = \bar L_e\gamma^\mu\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}L_e \ ,$$
$$(L^\mu)^\dagger = \bar\Psi_{e,L}\gamma^\mu\Psi_{\nu_e,L} = \bar L_e\gamma^\mu\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}L_e \ .$$

[Similar expressions hold for the quark piece].

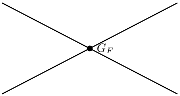
From Fermi model to $SU(2)_L$ gauge theory (1).

- **Problems of the Fermi model.**

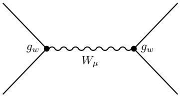
$$\mathcal{L}_{\rm Fermi} = -2\sqrt{2} G_F H_\mu L^\mu + \text{h.c.} \ .$$

  * Issues with quantum corrections, *i.e.*, **non-renormalizability**.
  * **Effective theory** valid up to an energy scale $E \ll m_w \approx 100$ GeV.
  * Fermi model is **not** based on **gauge symmetry principles**.

- **Solution: a gauge theory (Glashow, Salam, Weinberg, 60-70, [Nobel prize, 1979]).**
  * Four fermion interactions can be seen as a **s-channel diagram**.
  * Introduction of a **new gauge boson W_μ**.
  * This boson couples to fermions with a **strength g_w**.



$\Longrightarrow$



G_F

$$\frac{g_w^2}{p^2 - m_w^2} \approx -\frac{g_w^2}{m_w^2}$$

  * Prediction: $g_w \sim \mathcal{O}(1) \Rightarrow m_w \sim 100$ GeV.

From Fermi model to $SU(2)_L$ gauge theory (2).

- **Choice of the gauge group: suggested by the currents**:

$$L^\mu = \bar{L}_e \gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L_e = \bar{L}_e \gamma^\mu \frac{\sigma^1 + i\sigma^2}{2} L_e \ ,$$
$$(L^\mu)^\dagger = \bar{L}_e \gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} L_e . = \bar{L}_e \gamma^\mu \frac{\sigma^1 - i\sigma^2}{2} L_e \ .$$

[Similar expressions hold for the quark piece].

  - Two **Pauli matrices** appear naturally.
  - $\sigma^i/2$ are the generators of the $SU(2)$ algebra (in the **fundamental (dimension 2) representation**).

We choose the $SU(2)$ gauge group to describe weak interactions.

From Fermi model to $SU(2)_L$ gauge theory (3).

We choose the $SU(2)_L$ gauge group to describe weak interactions.

- $1/2\sigma^i$ are the generators of the **fundamental representation**.

$$\left[\frac{1}{2}\sigma^i, \frac{1}{2}\sigma^j\right] = i\epsilon^{ij}{}_k \frac{1}{2}\sigma_k \ ,$$

- The left-handed doublets lie in the **fundamental representation $\underset{\sim}{2}$**.
  - * The left-handed fields are the only ones **sensible** to weak interactions.
  - * A doublet is a **two-dimensional object**.
  - * The Pauli matrices are 2×2 matrices.
  - * This explains the L-subscript in $SU(2)_L$.
- The right-handed leptons lie in the **trivial representation $\underset{\sim}{1}$**.
  - * **Non-sensible** to weak interactions.
- $SU(2)_L \rightsquigarrow$ **three gauge bosons W^i_μ** with $i = 1, 2, 3$.

The $SU(2)_L$ gauge theory for weak interactions (1).

- **How to construct the $SU(2)_L$ Lagrangian?**
- **We start from the free Lagrangian for fermions**.
  * Simplification-1: **no quarks here**.
  * Simplification-2: **no right-handed neutrinos**.

$$\mathcal{L}_{\rm free} = \bar{L}_e\Big(i\gamma^\mu\partial_\mu\Big)L_e + \bar{e}_R\Big(i\gamma^\mu\partial_\mu\Big)e_R \ .$$

  * A mass term **mixes left and right-handed fermions**.
  * **The mass term are forbidden** since $L_e \sim \underset{\sim}{\mathbf{2}}$ and $e_R \sim \underset{\sim}{\mathbf{1}}$.
- **We make the Lagrangian invariant under $SU(2)_L$ gauge transformations**.
  * **$SU(2)_L$ gauge transformations** are given by

$$L_e \to \exp\left[ig_w\omega_i(x)\frac{\sigma^i}{2}\right]L_e = U(x)L_e \qquad \text{and} \qquad e_R \to e_R \ .$$

  * Gauge invariance requires **covariant derivatives**,

$$\partial_\mu L_e \to D_\mu L_e = \left[\partial_\mu - ig_w W_{\mu i}\frac{\sigma^i}{2}\right]L_e \qquad \text{and} \qquad \partial_\mu e_R \to D_\mu e_R = \partial_\mu e_R \ .$$

  * We have introduced one gauge boson for each generator $\Rightarrow$ **three $W_{\mu i}$**.

The $SU(2)_L$ gauge theory for weak interactions (2).

- **The matter sector Lagrangian reads then.**

$$\mathcal{L}_{\rm weak,matter} = \bar{L}_e\Big(i\gamma^\mu D_\mu\Big)L_e + \bar{e}_R\Big(i\gamma^\mu D_\mu\Big)e_R \ .$$

with

$$D_\mu L_e = \Big[\partial_\mu - ig_w W_{\mu i}\frac{\sigma^i}{2}\Big]L_e \qquad \text{and} \qquad D_\mu e_R = \partial_\mu e_R \ .$$

- **We must then add kinetic terms for the gauge bosons**:

$$\mathcal{L}_{\rm weak,gauge} = -\frac{1}{4}W^i_{\mu\nu}W_i^{\mu\nu} \ .$$

  * **The field strength tensor** reads:

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_w \epsilon^i{}_{jk} W^j_\mu W^k_\nu \ .$$

  * Gauge invariance implies the **transformation laws**:

$$\frac{\sigma^i}{2}W_i^\mu \to U\Big[\frac{\sigma^i}{2}W_i^\mu + \frac{i}{g_w}\partial^\mu\Big]U^\dagger \ .$$

The $SU(2)_L$ gauge theory for weak interactions (3).

The weak interaction Lagrangian for leptons.

$$\mathcal{L}_{\rm weak,e} = \bar{\mathbf{L}}_{\mathbf{e}}\Big(\mathbf{i}\gamma^\mu \mathbf{D}_\mu\Big)\mathbf{L}_{\mathbf{e}} + \bar{\mathbf{e}}_{\mathbf{R}}\Big(\mathbf{i}\gamma^\mu \mathbf{D}_\mu\Big)\mathbf{e}_{\mathbf{R}} - \frac{\mathbf{1}}{\mathbf{4}}\mathbf{W}^{\mathbf{i}}_{\mu\nu}\mathbf{W}_{\mathbf{i}}^{\mu\nu} \ .$$

with

$$D_\mu L_e = \Big[\partial_\mu - i g_w W_{\mu i}\frac{\sigma^i}{2}\Big] L_e \ ,$$

$$D_\mu e_R = \partial_\mu e_R \ ,$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_w \epsilon^i{}_{jk} W^j_\mu W^k_\nu \ .$$

- **Observation of the weak W^i_μ-bosons**:
  * The **experimentally observed $W^\pm$-bosons** are defined by

$$W^\pm_\mu = \frac{1}{2}\big(W^1_\mu \mp i W^2_\mu\big) \ .$$

  * **The W^3-boson cannot be identified to the Z^0 or γ**: Both couple to left-handed and right-handed leptons.

$SU(2)_L$ gauge theory cannot explain all data...

Summary - A gauge theory for weak interactions.

A gauge theory for weak interactions.

- Based on the **non-Abelian $SU(2)_L$** gauge group.
- Matter (1): **doublets with the left-handed component of the fields**.
  - **Fundamental representation**.
  - **Generators**: Pauli matrices (over two).
- Matter (2): the right-handed component of the fields are **singlet**.
- **Three massless gauge bosons**.
  - $(W^1_\mu, W^2_\mu) \Longrightarrow (W^+_\mu, W^-_\mu)$.
  - $W^3_\mu \neq Z^0_\mu, A_\mu \Rightarrow$ **need for another theory: the electroweak theory**.

Outline.

The electroweak theory (1).

- **Electromagnetism and weak interactions**:
  * $SU(2)_L$: **what is the neutral boson W^3**?
  * How to get a **single formalism** for electromagnetic and weak interactions?
- **Idea: introduction of the hypercharge Abelian group**:
  * $U(1)_Y$: we have a **neutral gauge boson** $B \Rightarrow B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.
  * $U(1)_Y$: we have a **coupling constant** g_Y.
  * $SU(2)_L \times U(1)_Y$: W^3 and B **mix** to the Z^0-boson and the photon.
- **Quantum numbers under the electroweak gauge group**:
  * $SU(2)_L$: left-handed quarks and leptons $\Rightarrow \underset{\sim}{\mathbf{2}}$.
  * $SU(2)_L$: right-handed quarks and leptons $\Rightarrow \underset{\sim}{\mathbf{1}}$.
  * $U(1)_Y$: fixed in order to reproduce the correct electric charges.

The electroweak theory (2).

Nœther procedure leads to the following Lagrangian.

$$\begin{aligned}\mathcal{L}_{\rm EW} = & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W_i^{\mu\nu} \\ & + \sum_{f=1}^{3}\Big[\bar{L}_f\Big(i\gamma^\mu D_\mu\Big)L^f + \bar{e}_{Rf}\Big(i\gamma^\mu D_\mu\Big)e_R^f\Big] \\ & + \sum_{f=1}^{3}\Big[\bar{Q}_f\Big(i\gamma^\mu D_\mu\Big)Q^f + \bar{u}_{Rf}\Big(i\gamma^\mu D_\mu\Big)u_R^f + \bar{d}_{Rf}\Big(i\gamma^\mu D_\mu\Big)d_R^f\Big] \ .\end{aligned}$$

* We have introduced the **left-handed lepton and quark doublets**

$$L^1 = \begin{pmatrix}\Psi_{\nu_e,L}\\ \Psi_{e,L}\end{pmatrix} \ , \qquad L^2 = \begin{pmatrix}\Psi_{\nu_\mu,L}\\ \Psi_{\mu,L}\end{pmatrix} \ , \qquad L^3 = \begin{pmatrix}\Psi_{\nu_\tau,L}\\ \Psi_{\tau,L}\end{pmatrix} \ ,$$
$$Q^1 = \begin{pmatrix}\Psi_{u,L}\\ \Psi_{d,L}\end{pmatrix} \ , \qquad Q^2 = \begin{pmatrix}\Psi_{c,L}\\ \Psi_{s,L}\end{pmatrix} \ , \qquad Q^3 = \begin{pmatrix}\Psi_{t,L}\\ \Psi_{b,L}\end{pmatrix} \ .$$

* We have introduced the **right-handed lepton and quark singlets**

$$e_R^1 = \Psi_{e,R} \ , \quad e_R^2 = \Psi_{\mu,R} \ , \quad e_R^3 = \Psi_{\tau,R} \ ,$$
$$u_R^1 = \Psi_{u,R} \ , \quad u_R^2 = \Psi_{c,R} \ , \quad u_R^3 = \Psi_{t,R} \ , \quad d_R^1 = \Psi_{d,R} \ , \quad d_R^2 = \Psi_{s,R} \ , \quad d_R^3 = \Psi_{b,R} \ .$$

The electroweak theory (3).

Nœther procedure leads to the following Lagrangian.

$$\begin{aligned}\mathcal{L}_{\rm EW} = & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W_i^{\mu\nu}\\ & + \sum_{f=1}^{3}\Big[\bar{L}_f\Big(i\gamma^\mu D_\mu\Big)L^f + \bar{e}_{Rf}\Big(i\gamma^\mu D_\mu\Big)e^f_R\Big]\\ & + \sum_{f=1}^{3}\Big[\bar{Q}_f\Big(i\gamma^\mu D_\mu\Big)Q^f + \bar{u}_{Rf}\Big(i\gamma^\mu D_\mu\Big)u^f_R + \bar{d}_{Rf}\Big(i\gamma^\mu D_\mu\Big)d^f_R\Big] \ .\end{aligned}$$

* **The covariant derivatives are given by**

$$D_\mu = \partial_\mu - ig_Y \mathbf{Y} B_\mu - ig_w \mathbf{T^i} W_{\mu i}$$

- ◇ **Y** is the hypercharge operator (to be defined).
- ◇ The representation matrices $\mathbf{T^i}$ are $\frac{\sigma^i}{2}$ and 0 for doublets and singlets.

Gauge boson mixing.

- **The neutral gauge bosons mix as**

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix} .$$

where the weak mixing angle θ_w will be defined later [see below...].

- **The neutral interactions (for the electron) are given by**

$$\begin{aligned}\mathcal{L}_{\rm int} &= \bar{L}_e\gamma^\mu\Big(g_Y Y_{L_e} B_\mu + g_w\frac{\sigma^3}{2}W_{\mu 3}\Big)L_e + \bar{e}_{Rf}\gamma^\mu g_Y Y_{e_R} B_\mu e_{Rf} \\ &= \bar{L}_e\gamma^\mu\Big(\cos\theta_w g_Y Y_{L_e} + \sin\theta_w g_w\frac{\sigma^3}{2}\Big)A_\mu L_e + \bar{e}_{Rf}\gamma^\mu\cos\theta_w g_Y Y_{e_R} A_\mu e_{Rf} \\ &+ \bar{L}_e\gamma^\mu\Big(-\sin\theta_w g_Y Y_{L_e} + \cos\theta_w g_w\frac{\sigma^3}{2}\Big)Z_\mu L_e - \bar{e}_{Rf}\gamma^\mu\sin\theta_w g_Y Y_{e_R} Z_\mu e_{Rf} \ .\end{aligned}$$

- **To reproduce electromagnetic interactions, we need**

$$\mathbf{e} = \mathbf{g_Y}\cos\theta_\mathbf{w} = \mathbf{g_w}\sin\theta_\mathbf{w} \quad \text{and} \quad \mathbf{Q} = \mathbf{Y} + \mathbf{T^3} \ .$$

This defines the hypercharge quantum numbers.

Field content of the electroweak theory.

Field	$SU(2)_L$ rep.	Quantum numbers Y	T^3	Q
$L^f = \begin{pmatrix} \Psi_{\nu_{e_f},L} \\ \Psi_{e_f,L} \end{pmatrix}$	$\underset{\sim}{2}$	$-\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	0 -1
e_R^f	$\underset{\sim}{1}$	-1	0	-1
$Q^f = \begin{pmatrix} \Psi_{u_f,L} \\ \Psi_{d_f,L} \end{pmatrix}$	$\underset{\sim}{2}$	$\frac{1}{6}$ $\frac{1}{6}$	$\frac{1}{2}$ $-\frac{1}{2}$	$\frac{2}{3}$ $-\frac{1}{3}$
u_R^f	$\underset{\sim}{1}$	$\frac{2}{3}$	0	$\frac{2}{3}$
d_R^f	$\underset{\sim}{1}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$

Electroweak symmetry breaking (1).

- **The weak $W^{\pm}$-bosons and Z^0-bosons are observed as massive.**
  - **The electroweak symmetry must be broken.**
  - **The photon must stay massless.**
- **Breaking mechanism: we introduce a Higgs multiplet φ.**
  - We need to **break** $SU(2)_L \Rightarrow \varphi$ cannot be an $SU(2)_L$-singlet.
  - The Z^0-boson is massive $\Rightarrow U(1)_Y$ **must be broken** $\Rightarrow Y_\varphi \neq 0$.
  - $U(1)_{e.m.}$ **is not broken** $\Rightarrow$ one component of φ is electrically neutral.
- **We introduce a Higgs doublet of $SU(2)_L$ with $Y_\varphi = 1/2$.**

$$\varphi = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \equiv \begin{pmatrix} h_1^+ \\ h_2^0 \end{pmatrix} .$$

- **The Higgs Lagrangian is given by** .

$$\mathcal{L}_{\rm Higgs} = D_\mu \varphi^\dagger \; D^\mu \varphi + \mu^2 \varphi^\dagger \varphi - \lambda \big(\varphi^\dagger \varphi\big)^2 = D_\mu \varphi^\dagger \; D^\mu \varphi - V(\varphi, \varphi^\dagger) \; .$$

  - The covariant derivative reads $D_\mu \varphi = \Big(\partial_\mu - \frac{i}{2} g_Y B_\mu - i g_w \frac{\sigma^i}{2} W_{\mu i}\Big) \varphi$.
  - **The scalar potential** is required for symmetry breaking.

Electroweak symmetry breaking (2).

- **At the minimum of the potential, the neutral component of φ gets a vev.**

$$\langle\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} .$$

- **We select the so-called unitary gauge**.

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h\end{pmatrix} .$$

* The three Goldstone bosons have been eliminated from the equations. They have been **eaten by the $W^{\pm}$ and Z^0 bosons** to get massive.
* The remaining degree of freedom is the **(Brout-Englert-)Higgs boson**.

Mass eigenstates - gauge boson masses (1).

We shift the scalar field by its vev.

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix} .$$

- **The Higgs covariant derivative reads then**:

$$D_\mu\varphi = \frac{1}{\sqrt{2}}\partial_\mu\begin{pmatrix} 0 \\ v+h \end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix} \frac{g_Y}{2}B_\mu + \frac{g_w}{2}W^3_\mu & \frac{g_w}{2}\left(W^1_\mu - iW^2_\mu\right) \\ \frac{g_w}{2}\left(W^1_\mu + iW^2_\mu\right) & \frac{g_Y}{2}B_\mu - \frac{g_w}{2}W^3_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v+h \end{pmatrix} .$$

- **From the kinetic terms, one obtains the mass matrix, in the $W^3 - B$ basis**.

$$D_\mu\varphi^\dagger\ D^\mu\varphi \to \begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix}\begin{pmatrix} \frac{1}{4}g_w^2v^2 & -\frac{1}{4}g_Yg_wv^2 \\ -\frac{1}{4}g_Yg_wv^2 & \frac{1}{4}g_Y^2v^2 \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} .$$

  * The physical states correspond to **eigenvectors** of the mass matrix.
  * The mass matrix is **diagonalized** after the rotation

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix} ,$$

with $\cos^2\theta_w = \frac{g_w^2}{g_w^2+g_Y^2}$.

Mass eigenstates - gauge boson masses (2).

- **The mass matrix is diagonalized after the rotation**

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} .$$

- **As for the weak theory, we rotate W_μ^1 and W_μ^2.**

$$W_\mu^\pm = \frac{1}{2}\big(W_\mu^1 \mp i W_\mu^2\big) .$$

- **After the two rotations, the Lagrangian reads**

$$D_\mu\varphi^\dagger \ D^\mu\varphi = \frac{e^2 v^2}{4\sin^2\theta_w} W_\mu^+ W^{-\mu} + \frac{e^2 v^2}{8\sin^2\theta_w \cos^2\theta_w} Z_\mu Z^\mu + \ldots \ .$$

  * We obtain **a $W^\pm$-boson mass term**, $m_w = \frac{ev}{2\sin\theta_w}$.
  * We obtain **a Z^0-boson mass term**, $m_z = \frac{ev}{2\sin\theta_w\cos\theta_w}$.
  * **The photon remains massless,** $m_\gamma = 0$.

Mass eigenstates - Higgs kinetic and interaction terms.

- **The Higgs kinetic and gauge interaction terms lead to**

$$D_\mu\varphi^\dagger\ D^\mu\varphi = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{e^2 v^2}{4\sin^2\theta_w} W^+_\mu W^{-\mu} + \frac{e^2 v^2}{8\sin^2\theta_w \cos^2\theta_w} Z_\mu Z^\mu$$
$$+ \frac{e^2 v}{2\sin^2\theta_w} W^+_\mu W^{-\mu} h + \frac{e^2 v}{4\sin^2\theta_w \cos^2\theta_w} Z_\mu Z^\mu h$$
$$+ \frac{e^2}{4\sin^2\theta_w} W^+_\mu W^{-\mu} hh + \frac{e^2}{8\sin^2\theta_w \cos^2\theta_w} Z_\mu Z^\mu hh \ .$$

* We obtain **gauge boson mass terms**.
* We obtain a Higgs **kinetic term**.
* We obtain **trilinear interaction terms**.
* We obtain **quartic interaction terms**.
* Remark: no interaction between the Higgs boson and the photon.

Mass eigenstates - fermion masses (1).

- **The fermion masses are obtained from the Yukawa interactions.**

$$\mathcal{L}_{\rm Yuk} = -\bar{u}_R y_u \big(Q\cdot\varphi\big) - \bar{d}_R y_d \big(\varphi^\dagger Q\big) - \bar{e}_R y_e \big(\varphi^\dagger L\big) + \text{h.c.}$$

  * We have introduced the $SU(2)$ invariant product $A\cdot B = A_1 B_2 - A_2 B_1$.
  * Flavor (or generation) indices are understood:

$$\bar{d}_R y_d \big(\varphi^\dagger Q\big) \equiv \sum_{f,f'=1}^{3} \bar{d}_{Rf'}\ \left(y_d\right)^{f'}{}_f\ \left(\varphi^\dagger Q^f\right)\ .$$

  * The Lagrangian terms are **matrix products in flavor space**.

- **The mass matrices read**

$$\mathcal{L}_{\rm mass} = -\frac{v}{\sqrt{2}}\bar{u}_R y_u u_L - \frac{v}{\sqrt{2}}\bar{d}_R y_d d_L - \frac{v}{\sqrt{2}}\bar{e}_R y_e e_L + \text{h.c.}\ ,$$

where we have performed the shift $\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h\end{pmatrix}$,
and introduced $u_L^f = \Psi_{u_f,L}, \ldots$.

Mass eigenstates - fermion masses (2).

- **The fermion mass Lagrangian read**:

$$\mathcal{L}_{\rm mass} = -\frac{v}{\sqrt{2}} \bar{u}_R y_u u_L - \frac{v}{\sqrt{2}} \bar{d}_R y_d d_L - \frac{v}{\sqrt{2}} \bar{e}_R y_e e_L + \text{h.c.} \ .$$

  * The physical states correspond to **eigenvectors** of the mass matrices.
  * Diagonalization: **any complex matrix** fulfill

$$y = V_R \ \tilde{y} \ U_L^\dagger \ ,$$

  with $\tilde{y}$ **real and diagonal and U_L, V_R unitary**.

- **Diagonalization of the fermion sector**: we got replacement rules,

$$u_L = \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} \to U_L^u u'_L \ , \qquad \bar{u}_R = \begin{pmatrix} \bar{u}_R & \bar{c}_R & \bar{t}_R \end{pmatrix} \to \bar{u}'_R \ (V_R^u)^\dagger \ , \qquad \ldots$$

  * **The up-type quark mass terms become**

$$-\frac{v}{\sqrt{2}} \bar{u}_R y_u u_L \to -\frac{v}{\sqrt{2}} \Big[\bar{u}'_R (V_R^u)^\dagger \Big] \Big[V_R^u \tilde{y}_u (U_L^u)^\dagger \Big] \Big[U_L^u u'_L \Big] = -\frac{v}{\sqrt{2}} \bar{u}'_R \tilde{y}_u u'_L$$

  where u_L, u_R are gauge-eigenstates and u'_L, u'_R mass-eigenstates.

Mass eigenstates - flavor and CP violation.

- **The neutral interactions are still diagonal in flavor space, *e.g.*,**

$$\mathcal{L}_{\rm int} = \frac{2}{3}\ e\ \bar{u}_L \gamma^\mu A_\mu u_L \to \frac{2}{3}\ e\ \Big[\bar{u}'_L (U^u_L)^\dagger\Big] \gamma^\mu A_\mu \Big[U^u_L u'_L\Big] = \frac{2}{3}\ e\ \bar{u}'_L \gamma^\mu A_\mu u'_L\ .$$

due to unitarity of U^u_L.

- **The charged interactions are now non-diagonal in flavor space, *e.g.*,**

$$\begin{aligned}\mathcal{L}_{\rm int} = \frac{e}{\sqrt{2}\sin\theta_w} \bar{u}_L \gamma^\mu W^+_\mu d_L &\to \frac{e}{\sqrt{2}\sin\theta_w} \Big[\bar{u}'_L (U^u_L)^\dagger\Big] \gamma^\mu W^+_\mu \Big[U^d_L d'_L\Big] \\ &= \frac{e}{\sqrt{2}\sin\theta_w} \bar{u}'_L \Big[(U^u_L)^\dagger U^d_L\Big] \gamma^\mu W^+_\mu d'_L\ .\end{aligned}$$

  * **Charged current interactions become proportionnal to the CKM matrix**,

$$\mathbf{V_{CKM}} = (\mathbf{U^u_L})^\dagger \mathbf{U^d_L} \qquad \text{[Nobel prize, 2008]}\ .$$

  * **One phase and three angles** to parameterize a unitary 3×3 matrix.
  $\Rightarrow$ **Flavor and CP violation in the Standard Model**.

Summary - The electroweak theory.

The electroweak theory.

- Based on the $SU(2)_L \times U(1)_Y$ gauge group.
  - * $SU(2)_L$**:** weak interactions, three W^i-bosons acting on left-handed fermions and on the Higgs field.
  - * $U(1)_Y$**:** hypercharge interactions, one B-boson acting on both left- and right-handed fermions and on the Higgs field.
- The gauge group is **broken to** $U(1)_{e.m.}$.
  - * The neutral component of the Higgs doublet **gets a vev**.
  - * **Hypercharge quantum numbers** are chosen consistently.
    $\Rightarrow$ The fields get the correct electric charge ($Q = T^3 + Y$).
  - * W^1 **and** W^2 **bosons mix to** $W^\pm$.
  - * B **and** W^3 **bosons mix to** Z^0 **and** γ.
- Yukawa interactions with the Higgs field lead to **fermion masses**.
- Experimental challenge: the **discovery of the Higgs boson**.

Outline.

The $SU(3)_c$ gauge group.

- **Discovery of the color quantum numbers.** **[Barnes *et al.* (1964)]**
  - The predicted $|\Omega\rangle = |sss\rangle$ baryon is a spin 3/2 particle.
  - The $|\Omega\rangle$ wave function is **fully symmetric** (spin + flavor).
  - **This contradicts the spin-statistics theorem**.

Introduction of the color quantum number.

- **The $SU(3)_c$ gauge group.**
  - Observed particles are **color neutral**.
  - The **minimal** way to write an **antisymmetric wave function** for $|\Omega\rangle$ is

$$|\Omega\rangle = \epsilon_{mn\ell}|s^m s^n s^\ell\rangle \ .$$

  - The quarks lie thus in a $\underset{\sim}{3}$ of the new gauge group $\Rightarrow$ $\mathbf{SU(3)_c}$.

Field content of the Standard Model and representation.

Field	$SU(3)_c$ rep.	$SU(2)_L$ rep.	$U(1)_Y$ charge
L_f	$\underset{\sim}{\mathbf{1}}$	$\underset{\sim}{\mathbf{2}}$	$-\frac{1}{2}$
e_{Rf}	$\underset{\sim}{\mathbf{1}}$	$\underset{\sim}{\mathbf{1}}$	-1
Q_f	$\underset{\sim}{\mathbf{3}}$	$\underset{\sim}{\mathbf{2}}$	$\frac{1}{6}$
u_{Rf}	$\underset{\sim}{\mathbf{3}}$	$\underset{\sim}{\mathbf{1}}$	$\frac{2}{3}$
d_{Rf}	$\underset{\sim}{\mathbf{3}}$	$\underset{\sim}{\mathbf{1}}$	$-\frac{1}{3}$
φ	$\underset{\sim}{\mathbf{1}}$	$\underset{\sim}{\mathbf{2}}$	$\frac{1}{2}$
B	$\underset{\sim}{\mathbf{1}}$	$\underset{\sim}{\mathbf{1}}$	0
W	$\underset{\sim}{\mathbf{1}}$	$\underset{\sim}{\mathbf{3}}$	0
g	$\underset{\sim}{\mathbf{8}}$	$\underset{\sim}{\mathbf{1}}$	0

* The matter Lagrangian involves the covariant derivative

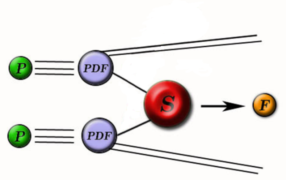
$$D_\mu = \partial_\mu - ig_Y Y B_\mu - ig_w T^i W_{\mu i} - ig_s \mathcal{T}^a g_{\mu a} \ .$$

* We introduce a kinetic term for each gauge boson.
* The Higgs potential and Yukawa interactions are as in the electroweak theory.

Asymptotic freedom & confinement.

- **Based on the color $SU(3)_c$ interactions.**
  * **The partons** (quarks, antiquarks, gluons) are **colored**.
  * Observable particles (mesons $q\bar{q}'$, baryons $qq'q''$) are **color neutral**.
- **Running coupling constants.**
  * **Weak and electromagnetic**: stronger at **high** energies (small distances).
  * **Strong**: stronger at **low** energies (large distances).
- **High energies: asymptotic freedom** **[Gross, Politzer, Wilczek, Nobel prize 2004]**.
  * **Small** value for $g_s \Rightarrow$ **perturbative calculations** as series of g_s.
  * The partons behave as **free particles**.
  * In hadron collisions at high energies (*e.g.*, LHC):
    Partons are interacting, not hadrons.
- **Low energies: confinement**.
  * **Large** value of $g_s \Rightarrow$ partons are **confined** into hadrons.
  * Non-perturbative physics $\Rightarrow$ **models, experimental fits, . . .**.
  * **Hadronization** at low energies $\Rightarrow$ jets.
  * In hadron collisions at high energies (*e.g.*, LHC):
    One **observes hadrons**, not partons.

Hadron collisions - QCD factorization theorem (1)



- **Quarks and gluons are not seen in Nature due to confinement**.
- **In a hadron collision at high energy, they do interact (asymptotic freedom)**.
- **Predictions can be made thanks to the QCD factorization theorem.**

$$\frac{\mathrm{d}\sigma_{\mathrm{hadr}}}{\mathrm{d}\omega} = \sum_{ab} \int \mathrm{d}x_a\, \mathrm{d}x_b\, f_{a/A}(x_a; \mu_F)\, f_{b/B}(x_b; \mu_F) \frac{\mathrm{d}\sigma_{\mathrm{part}}}{\mathrm{d}\omega}(x_a, x_b, p_a, p_b, \ldots, \mu_F) \ ,$$

where σ_{hadr} is the hadronic cross section (hadrons $\to$ any final state).

* $\sum_{ab} \Rightarrow$ **all** partonic initial states (partons $a, b = q, \bar{q}, g$).
* x_a is the **momentum fraction** of the hadron A carried by the parton a.
* x_b is the **momentum fraction** of the hadron B carried by the parton b.
* If the final state contains any parton:
  Fragmentation functions (**from partons to observable hadrons**).

Hadron collision - QCD factorization theorem (2)

- **Predictions can be made thanks to the QCD factorization theorem.**

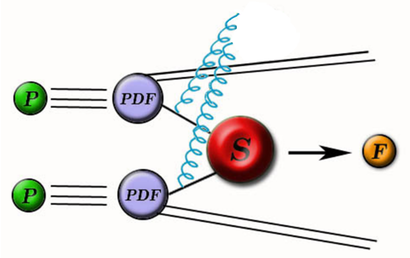
$$\frac{\mathrm{d}\sigma_{\mathrm{hadr}}}{\mathrm{d}\omega} = \sum_{ab} \int \mathrm{d}x_a \, \mathrm{d}x_b \, f_{a/A}(x_a; \mu_F) \, f_{b/B}(x_b; \mu_F) \frac{\mathrm{d}\sigma_{\mathrm{part}}}{\mathrm{d}\omega}(x_a, x_b, p_a, p_b, \ldots, \mu_F) \ .$$

- * $f_{a/p_1}(x_a; \mu_F), f_{b/p_2}(x_b; \mu_F)$: **parton densities**.
  - ◇ **Long distance physics**.
  - ◇ 'Probability' to have a parton with a momentum fraction x in a hadron.
- * $\mathrm{d}\sigma_{\mathrm{part}}$: **differential partonic cross section** (which you can now calculate).
  - ◇ **Short distance physics**.

μ_F - **Factorization scale**.
(how to distinguish long and short distance physics).

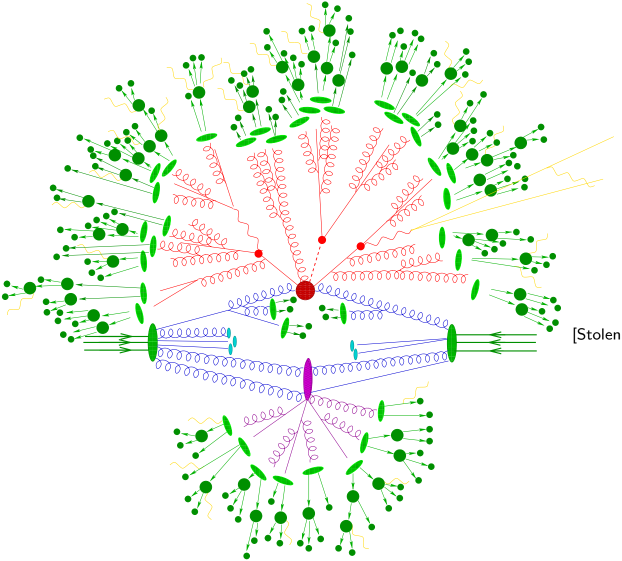
Parton showering and hadronization

* At high energy, initial and final state partons **radiate other partons**.



* Finally, very low energy partons **hadronize**.

Summary - The real life of a collision at the LHC

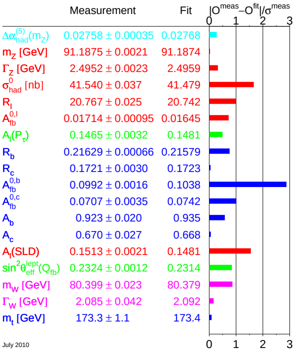


[Stolen from SHERPA]

Outline.

The Standard Model: advantages and open questions (1).

- **The Standard Model of particle physics.**
  - * Is a **mathematically consistent** theory.
  - * Is **compatible** with (almost) all experimental results [*e.g.*, LEP EWWG].



The Standard Model: advantages and open questions (2).

- **Open questions.**
  - * Why are there **three families** of quarks and leptons?
  - * Why does one family consist of $\{\mathbf{Q}, \mathbf{u_R}, \mathbf{d_R}; \mathbf{L}, \mathbf{e_r}\}$?
  - * Why is the **electric charge quantized**?
  - * Why is the local gauge group $\mathbf{SU(3)_c} \times \mathbf{SU(2)_L} \times \mathbf{U(1)_Y}$?
  - * Why is the spacetime **four-dimensional**?
  - * Why is there **26 free parameters**?
  - * What is the **origin of quark and lepton masses and mixings**?
  - * What is the **origin of CP violation**?
  - * What is the **origin of matter-antimatter asymmetry**?
  - * What is the **nature of dark matter**?
  - * What is the **role of gravity**?
  - * Why is the **electroweak scale (100 GeV) much lower than the Planck scale** (10^{19} **GeV)**?

Outline.

Grand Unified Theories: a unified gauge group (1).

- **Can we reduce the arbitrariness in the Standard Model?**
  * A **single direct factor** for the gauge group.
  * A **common representation** for quarks and leptons.
  * **Unification of g_Y, g_w and g_s** to a single coupling constant.

- **Unification of the Standard Model coupling constants.**
  * The coupling constants (at zero energy) **are highly different**.

Electromagnetism	Weak	Strong
$\sim 1/137$	$\sim 1/30$	~ 1

  * The coupling strengths depend on the energy due to **quantum corrections**.

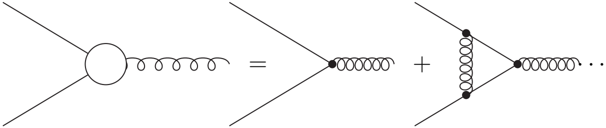
Unification.

◇ There exists a **unification scale**.

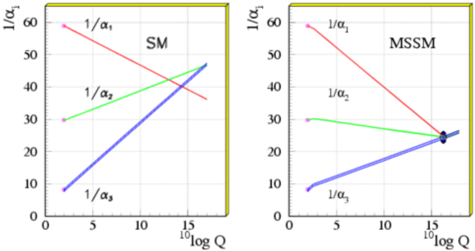
◇ The coupling strengths are **identical**.

Grand Unified Theories: a unified gauge group (2).

- **Running of the coupling constants.**
  * The coupling constant at first order of perturbation theory reads



  * These calculations lead to the **energy dependence** of the couplings.



  * Unification requires **additional matter** (*e.g.*, supersymmetry [see below...]).

Grand Unified Theories: a unified gauge group (3).

- **Construction of a Grand Unified Toy Theory.**
- **How to choose of a grand unified gauge group.**
  * We want to pick up G so that $\mathbf{SU(3)_c \times SU(2)_L \times U(1)_Y \subset G}$.
  * **Electromagnetism must not be broken**.
  * **The Standard Model must be reproduced at low energy**.
  * Matter must be **chiral**.
  * Interesting cases are:

$$G = \begin{cases} SU(N) & \text{with } N > 4 \\ SO(4N+2) & \text{with } N \geq 2 \\ E_6 & \end{cases}$$

- **How to specify representations for the matter fields.**
  * **The Standard Model must be reproduced at low energy**.
  * The choice for the Higgs fields $\Leftrightarrow$ **breaking mechanism**.
- **Specify the Lagrangian.**

$$\mathcal{L}_{\rm GUT} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm Yuk} + \mathcal{L}_{\rm breaking} \ .$$

Grand Unified Theories: a unified gauge group (4).

$$\mathcal{L}_{\rm GUT} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm Yuk} + \mathcal{L}_{\rm breaking}\ .$$

- $\mathcal{L}_{\rm kin}$: **Poincaré invariance**.
- $\mathcal{L}_{\rm kin} + \mathcal{L}_{\rm gauge}$: **gauge invariance**.
- $\mathcal{L}_{\rm Yuk}$: **Yukawa interactions** between Higgs bosons and fermions.
  - Must be **gauge-invariant**.
  - Fermion **masses** after symmetry breaking.
  - Flavor and CP violation.
  - **Not obtained (in general) from symmetry principles**.
- $\mathcal{L}_{\rm breaking}$: less known...

Grand Unified Theories: example of $SU(5)$ (1).

- **Gauge bosons**
  - * A 5×5 matrix contains naturally $SU(3)$ and $SU(2)$.

$$\begin{pmatrix} SU(3) & LQ \\ LQ^\dagger & SU(2) \end{pmatrix} \in SU(5)$$

  - * We have **12 additional gauge bosons**, the so-called leptoquarks (LQ).
  - * The matrix is traceless.
    - $\rightsquigarrow$ The hypercharge is quantized.
    - $\rightsquigarrow$ **The electric charge is quantized** ($Q = T^3 + Y$).

$$Y = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \rightsquigarrow Q = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

  - * This matches the quantum numbers of the **right-handed down antiquark** d_R^c and the **left-handed lepton doublet** L.

Grand Unified Theories: example of $SU(5)$ (2).

- **Fermions**
  - * **Fundamental representation** of $SU(5)$: d_R^c and L.
  - * $\underset{\sim}{\mathbf{10}}$ representation (antisymmetric matrix) $\equiv$ 10 degrees of freedom.
    $\rightsquigarrow$ the rest of the matter fields (10 degrees of freedom).

$$\underset{\sim}{\mathbf{5}} \equiv \begin{pmatrix} d_R^c \\ L \end{pmatrix} = \begin{pmatrix} (d_R^c)_r \\ (d_R^c)_g \\ (d_R^c)_b \\ \nu_L \\ e_L \end{pmatrix} \qquad \underset{\sim}{\mathbf{10}} \equiv \begin{pmatrix} 0 & (u_R^c)_b & -(u_R^c)_g & -(u_L)_r & -(d_L)_r \\ -(u_R^c)_b & 0 & (u_R^c)_r & -(u_L)_g & -(d_L)_g \\ (u_R^c)_g & -(u_R^c)_r & 0 & -(u_L)_b & -(d_L)_b \\ (u_L)_r & (u_L)_g & (u_L)_b & 0 & -e_R^c \\ (d_L)_r & (d_L)_g & (d_L)_b & e_R^c & 0 \end{pmatrix}$$

  - * The embedding of the gauge boson into $SU(5)$ is **easy**.
  - * The embedding of the fermion sector is **miraculous**.

Grand Unified Theories: example of $SU(5)$ (3).

- **Higgs sector**
  - * **Two Higgs fields are needed**.
    - ◇ One to break $SU(5) \to SU(3)_c \times SU(2)_L \times U(1)_Y$.
    - ◇ One to break $SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_{e.m.}$.
  - * The **simplest choice**.
    - ◇ One field in the $\underset{\sim}{\mathbf{24}}$ representation (special unitary 5×5 matrix).
    - ◇ One field in the fundamental representation.

- **Advantages of $SU(5)$**
  - * **Unification of all the interactions** within a simple gauge group.
  - * **Partial unification of the matter** within two multiplets.
  - * **Electric charge quantization**.

- **Problems specific to $SU(5)$**
  - * Prediction of the **proton decay** (lifetime: $10^{31} - 10^{33}$ years).
  - * Prediction of a **magnetic monopole**.
  - * Other problems shared with the Standard Model (three families, *etc.*)

Grand Unified Theories: $SO(10)$, E_6.

- **Matter content**.
  - * The matter is unified within a **single multiplet**.
  - * $SO(10)$ has an additional degree of freedom $\Rightarrow$ the right-handed neutrino.
  - * Explanation for the **neutrino masses**.
  - * E_6 contains several additional degrees of freedom
    $\Rightarrow$ **the right-handed neutrino plus new particles** (to be discovered...).

- **The breaking mechanism leads to additional $U(1)$ symmetrie(s)**.
  - * The gauge boson(s) associated to these new $U(1)$ are called **Z′** bosons.
  - * Massive Z' **resonances** are searched at colliders [see exercises classes].

$$pp \to \gamma, Z, Z' + X \to e^+e^- + X \text{ or } \mu^+\mu^- + X \ .$$

- **Other specific advantages and problems**.
  - * E_6 appears naturally in string theories.
  - * There is still no explanation for, *e.g.*, the number of families.
  - * **Gauge coupling unification is impossible without additional matter**
    $\Rightarrow$ *e.g.*, supersymmetry.

Summary - Grand Unified Theories.

Grand Unified Theories.

- The Standard model gauge group, $SU(3)_c \times SU(2)_L \times U(1)_Y$, is embedded in a **unified gauge group**.
- **Common representations** are used for quarks and leptons.
- **The Standard Model is reproduced at low energy**.
- More or less complicated **breaking mechanism**.
- Examples: $SU(5)$, $SO(10)$, E_6, ...

Outline.

Supersymmetry: Poincaré superalgebra (1).

Invariant Lagrangians Symmetries	$\Longrightarrow$ Nœther theorem	**Conserved charges**
$\Downarrow$ Fock spaces		$\Downarrow$ Quantization
Hilbert space	$\Longleftarrow$ Representations	**Symmetry generators**

- **Ingredients leading to superalgebra/supersymmetry**.
  - * We have two types of particles, **fermions and bosons**.
    $\Rightarrow$ We have **two types of conserved charges, B and F**.
  - * The composition of two symmetries is a symmetry.
    $\Rightarrow$ This imposes **relations between the conserved charges**.

$$\left[B_a, B_b\right] = i f_{ab}{}^c B_c \ ,$$

$$\left[B_a, F_i\right] = R_{ai}{}^j F_j \ ,$$

$$\left\{F_i, F_j\right\} = Q_{ij}{}^a B_a \ .$$

Supersymmetry: Poincaré superalgebra (2).

- **The Coleman-Mandula theorem (1967).**
  * The symmetry generators are assumed **bosonic**.
  * The only possible symmetry group in Nature is

$$G = \text{Poincaré} \quad \times \quad \text{gauge symmetries} \ .$$

  - **Spacetime symmetries**: Poincaré

$$\begin{aligned} \Big[L^{\mu\nu}, L^{\rho\sigma}\Big] &= -i\Big(\eta^{\nu\sigma} L^{\rho\mu} - \eta^{\mu\sigma} L^{\rho\nu} + \eta^{\nu\rho} L^{\mu\sigma} - \eta^{\mu\rho} L^{\nu\sigma}\Big) \ , \\ \Big[L^{\mu\nu}, P^{\rho}\Big] &= -i\Big(\eta^{\nu\rho} P^{\mu} - \eta^{\mu\rho} P^{\nu}\Big) \ , \\ \Big[P^{\mu}, P^{\nu}\Big] &= 0 \ , \end{aligned}$$

  - Internal gauge symmetries: compact **Lie algebra**.

$$\Big[T_a, T_b\Big] = i f_{ab}{}^c T_c \ , \qquad \Big[P_\mu, T_a\Big] = 0 \qquad \text{and} \qquad \Big[L_{\mu\nu}, T_a\Big] = 0 \ .$$

- **The Haag-Łopuszański-Sohnius theorem (1975).**
  * Extension of the Coleman-Mandula theorem.
  * **Fermionic generators** are included.
  * The minimal choice consists in a set of **Majorana spinors** $(Q, \overline{Q})$.
  * $N = 1$ **supersymmetry: one single supercharge** Q.

Supersymmetry: Poincaré superalgebra (3).

The Poincaré superalgebra.

- **Spacetime symmetries**.

$$\Big[L^{\mu\nu},L^{\rho\sigma}\Big]=-i\Big(\eta^{\nu\sigma}L^{\rho\mu}-\eta^{\mu\sigma}L^{\rho\nu}+\eta^{\nu\rho}L^{\mu\sigma}-\eta^{\mu\rho}L^{\nu\sigma}\Big)\ ,$$

$$\Big[L^{\mu\nu},P^{\rho}\Big]=-i\Big(\eta^{\nu\rho}P^{\mu}-\eta^{\mu\rho}P^{\nu}\Big)\ ,\qquad \Big[P^{\mu},P^{\nu}\Big]=0\ ,$$

- **Gauge symmetries**.

$$\Big[T_a,T_b\Big]=if_{ab}{}^{c}\,T_c\ ,\qquad \Big[P_\mu,T_a\Big]=0\qquad\text{and}\qquad\Big[L_{\mu\nu},T_a\Big]=0\ .$$

- **Supersymmetry**.

$$\Big[L^{\mu\nu},Q_\alpha\Big]=\big(\sigma^{\mu\nu}\big)_\alpha{}^\beta Q_\beta\qquad Q\text{ is a left-handed spinor}\ ,$$

$$\Big[L^{\mu\nu},\overline{Q}^{\dot\alpha}\Big]=\big(\bar\sigma^{\mu\nu}\big)^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta}\qquad \overline{Q}\text{ is a right-handed spinor}\ ,$$

$$\Big[Q_\alpha,P^\mu\Big]=\Big[\overline{Q}^{\dot\alpha},P^\mu\Big]=0\ ,$$

$$\Big\{Q_\alpha,\overline{Q}_{\dot\alpha}\Big\}=2\sigma^\mu{}_{\alpha\dot\alpha}P_\mu\ ,\qquad \Big\{Q_\alpha,Q_\beta\Big\}=\Big\{\overline{Q}^{\dot\alpha},\overline{Q}^{\dot\beta}\Big\}=0\ ,$$

$$\Big[Q_\alpha,T_a\Big]=\Big[Q_{\dot\alpha},T_a\Big]=0\qquad (Q,\overline{Q})\text{ is a gauge singlet}\ .$$

Supersymmetry: Poincaré superalgebra (4).

- **Consequences and advantages**.
  * The supercharge operators **change the spin** of the fields.

$$Q|\text{boson}\rangle = |\text{fermion}\rangle \qquad \text{and} \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle \ .$$

  * $(Q, \overline{Q})$ and P commute.
    $\Rightarrow$ fermions and bosons in a same multiplet have the **same mass**.

$$P^2|\text{boson}\rangle = m^2|\text{boson}\rangle \qquad \text{and} \qquad P^2|\text{fermion}\rangle = m^2|\text{fermion}\rangle \ .$$

  * The composition of two supersymmetry operations is a **translation**.

$$Q\overline{Q} + \overline{Q}Q \sim P \ .$$

  * **Scalar masses** are protected from quantum corrections.
  * It includes naturally **gravity**
    $\Rightarrow$ **New vision of spacetime** $\Rightarrow$ supergravity, superstrings.
  * **Unification** of the gauge coupling constants.

The Minimal Supersymmetric Standard Model.

- **Supersymmetry in particle physics**.
  * The Minimal Supersymmetric Standard Model: one **single supercharge**.
  * We associate **one (new) superpartner** to each Standard Model field.

    * Quarks $\Leftrightarrow$ squarks.
    * Leptons $\Leftrightarrow$ sleptons.
    * Gauge/Higgs bosons $\Leftrightarrow$ gauginos/higgsinos.

- **Supersymmetry breaking**.
  * No scalar electron has been discovered.
  * No massless photino has been observed.
  * *etc.*.

Supersymmetry has to be broken.

Supersymmetry breaking

- **No supersymmetry discovery until now.**
- **Supersymmetry breaking.**
  * **Superparticle masses shifted** to a higher scale.
  * **Breaking mechanism not fully satisfactory**.
  * Assumed to occur in a **hidden sector**.
  * **Mediated** through the visible sector via a given interaction.
  * **Examples**: minimal supergravity, gauge-mediated supersymmetry-breaking, *etc.*.
- **Form of a supersymmetric Lagrangian.**

$$\mathcal{L}_{\rm SUSY} = \mathcal{L}_{\rm matter} + \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm break} \ .$$

  * $\mathcal{L}_{\rm matter}$: supersymmetric gauge interactions for the matter sector.
  * $\mathcal{L}_{\rm gauge}$: supersymmetric gauge interactions for the gauge sector.
  * $\mathcal{L}_{\rm int}$: supersymmetric (non-gauge) interactions for the matter sector.
  * $\mathcal{L}_{\rm break}$: non-supersymmetric pieces of the Lagrangian (breaking).
    $\rightsquigarrow$ $\mathcal{O}(100)$ **new free parameters** $\Rightarrow$ **universality assumptions** $\rightsquigarrow \mathcal{O}(10)$.

Some supersymmetric phenomenology.

- ***R*-parity**.
  - $\mathcal{L}_{\rm int}$: Under its general form:
    $\leadsto$ **lepton number violating** interactions.
    $\leadsto$ **baryon number violating** interactions.
    $\leadsto$ **proton decay**.
  - Introduction of an *ad hoc* **discrete symmetry, *R*-parity**.
    - ◇ Standard Model fields: $R = +1$.
    - ◇ Superpartners: $R = -1$.
    - ◇ The problematic interactions **are forbidden**.
- **Consequences**.
  - **The lightest superpartner (LSP) is stable**.
    $\Rightarrow$ **Cosmology: must be neutral and color singlet.**
    $\Rightarrow$ **Possible dark matter candidate**.
  - Superparticles are produced **in pairs**.
    $\Rightarrow$ **Cascade-decays to the LSP**.
    $\Rightarrow$ **Missing energy collider signature**.

Summary - Supersymmetry.

Supersymmetry.

- Extension of the Poincaré algebra to the **Poincaré superalgebra**.
- Introduction of **supercharges**.
- The Minimal Supersymmetric Standard Model: **one single supercharge**.
  - **One superpartner** for each Standard Model field.
  - LSP: possible **dark matter candidate**.
  - LSP: Collider signatures with **large missing energy**.
- More or less complicated **breaking mechanism**.

Outline.

Extra-dimensions in a nutshell (1).

- **Main idea: the spacetime is not four-dimensional**.
- **Example: five dimensional scenario:** $\mathbb{R}^4 \times$**circle of radius** R.
  - The fifth dimension is **periodic**.
  - Massless 5D-fields $\Rightarrow$ **tower of 4D-fields**.

$$\phi(x^\mu, y) = \sum_n \phi_n(x^\mu) \exp\left[\frac{iny}{R}\right] ,$$

where y is the fifth-dimension corrdinate.
  - The 4D-fields ϕ_n are **massive**. Case of the scalar fields:
    - ◇ We start from the Klein-Gordon equation in 5D

$$\pentagon\phi(x^\mu, y) = [\Box - \partial_y^2]\phi(x^\mu, y) \Longrightarrow [\Box + \frac{n^2}{R^2}]\phi_n(x^\mu) = 0$$

  - No observation of a Kaluza-Klein excitation $(\phi_n) \Rightarrow$ **1/R must be large**.

Extra-dimensions in a nutshell (2).

- **Kaluza-Klein and unification**.
  - * Basic idea: **unification of electromagnetism and gravity** (20's).
  - * The 5D metric reads, with $M, N = 0, 1, 2, 3, 4$

$$g_{MN} \sim \begin{pmatrix} g_{\mu\nu} & A_\mu = g_{\mu 4} \\ A_\mu = g_{4\mu} & \phi = g_{44} \end{pmatrix} .$$

  - * **5D gravity $\rightsquigarrow$ 4D electromagnetism and gravity.**
- **Extension to all interactions**.
  - * The Standard Model needs **11** dimensions [Witten (1981)].
  - * **Problems with mirror fermions**.
- **One possible viable model: Randall-Sundrum (1999)**.
  - * The Standard Model fields lie on a **three-brane** (a 4D spacetime).
  - * Gravity lies in the bulk (all the 5D space).
  - * The size of the extra-dimensions can be **large** (TeV scale).
  - * *KK*-parity: dark matter candidate, missing energy signature, *etc.*
- **Other viable models are possible** (universal extra-dimensions, *etc.*).

Outline.

String theory in a nutshell.

- **Point-like particles ⇒ closed and/or open 1D-strings**.
- **String propagation in spacetime ⇒ a surface called a worldsheet**.
- **The vibrations of the string ⇒ elementary particles**.
- **Worldsheet physics ⇒ spacetime physics (quantum consistency)**.
  - * 10-dimensional spacetime (**extra-dimensions**).
  - * Extended gauge group (**Grand Unified Theories**).
  - * Supergravity (**supersymmetry**).
- **Compactification from 10D to 4D.**
  - * Must **reproduce the Standard Model**.
  - * **Many possible solutions**.
  - * **No solution found** so that all experimental constraints are satisfied.

Outline.

Summary.

- **The Standard Model has been constructed from experimental input**.
  * Based on symmetry principle (relativity, gauge invariance).
  * Is consistent with quantum mechanics.
  * Is the most tested theory of all time.
  * Suffers from some limitations and open questions.
- **Beyond the Standard Model theories are built from theoretical ideas**.
  * Ideas in constant evolution.
  * Grand Unified Theories.
  * Supersymmetry.
  * Extra-dimensions.
  * String theory.
  * *etc.*.